

## **On the Distribution of Estimated Technical Efficiency in Stochastic Frontier Models: Revisited**

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### **Abstract**

Both theoretical and empirical literatures on the stochastic frontier production model have been substantially enriched since 1977. However, attempts to derive the distribution of the estimated one-sided error term, even under restricted assumptions, have only recently been provided. This study attempts to derive the distribution of the estimated one-sided inefficiency error and several important statistical properties under fairly general assumptions. We also derive the distribution of estimated technical efficiency and provide an empirical illustration.

*Key words:* stochastic frontier; inefficiency error; estimated technical efficiency; leather industry

*JEL classification:* C34; D24

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### **1. Introduction**

The stochastic frontier production function (SFPF) model, a model to measure output-oriented technical efficiency (TE) of a firm, independently developed by Aigner et al. (1977) and Meeusen and van den Broeck (1977), has been widely used in the empirical parametric literature of production economics. The frontier is the locus of *maximum* levels of producible output with the help of the existing technology, at all feasible combinations of quantities of inputs. From its very name it is evident that the frontier itself is stochastic, in the sense that beside the usual one-sided error component introduced to capture the level of possible technical inefficiency of the producer, a two-sided stochastic term, introduced to capture the possible effects of all the random components beyond the control of the producer, can also affect the frontier level of output. Hence, individual production behavior is subject to a composite error term. To estimate such a model we, first of all, have to assume theoretical distributions for these two components of the composite error. It is common practice in the literature to assume that the two-sided random disturbance

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term is normally distributed white noise, i.e., it has zero mean and constant variance. On the other hand, several one-sided distributions have been used in the literature for the inefficiency error term (see for instance Coelli et al., 1998, chapters 8 and 9; Kumbhakar et al., 2000, chapter 3). It can be noted in this connection that the distribution of the estimated inefficiency term may not look like the one which, prior to estimation, is assumed. Unfortunately, no serious attempt has been provided to look into this issue until recently, when Wang and Schmidt (2009) theoretically derived the distribution of the estimated one-sided inefficiency error term, assuming it follows a half-normal distribution.

The objective of this paper is three-fold. First, we attempt to theoretically derive (in light of Wang and Schmidt, 2009) the probability density function of the estimated inefficiency error term, assuming that it has a general truncated normal distribution. We also state its several important statistical properties. Second, we derive the theoretical density of the estimated TE of firms. Finally, we plot this theoretical density using a real dataset and see whether the distribution of the estimated TE of the firms is consistent with this theoretical density.

The paper unfolds as follows. In Section 2, we derive the theoretical density of the estimated inefficiency and its important statistical properties. In this section we also derive the probability density of the estimated TE of the firms. In Section 3, we describe our empirical model, and Section 4 concludes.

## 2. Theoretical Density of the Estimated Technical Efficiency

Here we show the distribution of the estimated inefficiency error as well as estimated TE. We assume that  $u_i \sim iid N^+(\mu_i = z_i\delta, \sigma_u^2)$  and  $v_i \sim iid N(0, \sigma_v^2)$ .<sup>1</sup> Let the probability density function of any random variable  $\theta$  be denoted  $f_\theta$  and the joint density function of any two random variables  $\theta$  and  $\vartheta$  be denoted  $f_{\theta, \vartheta}$ . We use two standard results: (a) if  $m = m(p)$  where  $p$  is a random variable and thus  $m$  is also a random variable, then  $f_m = f_p |\partial p / \partial m|$  and (b) if  $q = q(x, y)$  and  $r = r(x, y)$ , then  $f_{q, r} = f_{x, y} |\partial(x, y) / \partial(q, r)|$ . The second term of the right-hand-side expression in either (a) and (b) is known as the Jacobian of the transformation. Observe that the term within “| |” in (a) is a scalar but in (b) is a matrix. Here,  $|\partial p / \partial m|$  refers to the absolute value of  $\partial p / \partial m$  and  $|\partial(x, y) / \partial(q, r)|$  refers to the determinant of:

$$\frac{\partial(x, y)}{\partial(q, r)} \equiv \begin{pmatrix} \left[ \begin{array}{cc} \frac{\partial x}{\partial q} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial q} & \frac{\partial y}{\partial r} \end{array} \right] \end{pmatrix}.$$

Considering the simple transformations  $z_1 = z_1(u, v) = u$  and  $z_2 = z_2(u, v) = \varepsilon = v - u$ , we get  $u = z_1$  and  $v = z_2 + z_1$ . Hence, using (b) we get:

$$f_{z_1, z_2} = f_{u, v} \left| \frac{\partial(u, v)}{\partial(z_1, z_2)} \right|,$$

where

$$\left| \frac{\partial(u, v)}{\partial(z_1, z_2)} \right| = \det \begin{bmatrix} \frac{\partial u}{\partial z_1} & \frac{\partial u}{\partial z_2} \\ \frac{\partial v}{\partial z_1} & \frac{\partial v}{\partial z_2} \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 1 - 0 = 1.$$

Hence,  $f_{u, \varepsilon}(u_i, \varepsilon_i) = f_{u, v}(u_i, v_i(u_i, \varepsilon_i))$ .

In order to simplify expressions we shall use the following notation and simplifications.

(i) For the distribution function of a standard normal variable,  $\Phi(\cdot)$ :

$$\left\{ 1 - \Phi\left(-\frac{z_i \delta}{\sigma_u}\right) \right\} = \Phi\left(\frac{z_i \delta}{\sigma_u}\right).$$

(ii) Define the following notations:

$$\mu_i^* = \frac{z_i \delta \sigma_v^2 - \varepsilon_i \sigma_u^2}{\sigma_u^2 + \sigma_v^2}; \quad \sigma_*^2 = \frac{\sigma_u^2 \sigma_v^2}{\sigma_u^2 + \sigma_v^2}; \quad \gamma = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2}.$$

(iii) Note that:

$$\left(\frac{z_i \delta}{\sigma_u}\right)^2 + \left(\frac{\varepsilon_i}{\sigma_v}\right)^2 - \left(\frac{\mu_i^*}{\sigma_*}\right)^2 = \left[\frac{z_i \delta + \varepsilon_i}{\sqrt{\sigma_u^2 + \sigma_v^2}}\right]^2.$$

(iv) Note that using (iii):

$$\begin{aligned} \left(\frac{u_i - z_i \delta}{\sigma_u}\right)^2 + \left(\frac{\varepsilon_i + u_i}{\sigma_v}\right)^2 &= \frac{u_i^2 - 2u_i z_i \delta + z_i^2 \delta^2}{\sigma_u^2} + \frac{\varepsilon_i^2 + 2\varepsilon_i u_i + u_i^2}{\sigma_v^2} \\ &= \left[\frac{z_i \delta + \varepsilon_i}{\sqrt{\sigma_u^2 + \sigma_v^2}}\right]^2 + \left(\frac{u_i - \mu_i^*}{\sigma_*}\right)^2. \end{aligned}$$

(v) Note that:

$$-u_i - \frac{1}{2} \left(\frac{u_i - \mu_i^*}{\sigma_*}\right)^2 = -\frac{1}{2} \left[\frac{u_i - (\mu_i^* - \sigma_*^2)}{\sigma_*}\right]^2 + \left(\frac{\sigma_*^2}{2} - \mu_i^*\right).$$

Using (i)–(iv), the joint density function of  $(u_i, \varepsilon_i)$  may be rewritten as

$$(vi) \quad f_{u, \varepsilon}(u_i, \varepsilon_i) = \frac{1}{\sigma_u \sigma_v (2\pi) \Phi\left(\frac{z_i \delta}{\sigma_u}\right)} \exp \left[ -\frac{1}{2} \left\{ \left(\frac{u_i - z_i \delta}{\sigma_u}\right)^2 + \left(\frac{\varepsilon_i + u_i}{\sigma_v}\right)^2 \right\} \right]$$

$$= \frac{1}{\sigma_u \sigma_v (2\Pi) \Phi\left(\frac{z_i \delta}{\sigma_u}\right)} \exp \left[ -\frac{1}{2} \left\{ \left[ \frac{z_i \delta + \varepsilon_i}{\sqrt{\sigma_u^2 + \sigma_v^2}} \right]^2 + \left( \frac{u_i - \mu_i^*}{\sigma_*} \right)^2 \right\} \right].$$

Therefore, the marginal density of  $\varepsilon$  is given by:

$$(vii) \quad f_\varepsilon(\varepsilon_i) = \int_0^\infty f_{u,\varepsilon}(u_i, \varepsilon_i) du_i = \frac{\exp \left[ -\frac{1}{2} \left\{ \left( \frac{z_i \delta + \varepsilon_i}{\sqrt{\sigma_u^2 + \sigma_v^2}} \right)^2 \right\} \right] \Phi\left(\frac{\mu_i^*}{\sigma_*}\right)}{\sqrt{2\Pi(\sigma_u^2 + \sigma_v^2)} \Phi\left(\frac{z_i \delta}{\sigma_u}\right)}.$$

$$(viii) \quad \text{Thus, } \hat{u}_i = E(u_i | \varepsilon_i) = \int_0^\infty u_i f_u(u_i | \varepsilon_i) du_i = \mu_i^* + \sigma_* \frac{\phi\left(-\frac{\mu_i^*}{\sigma_*}\right)}{\left[1 - \Phi\left(-\frac{\mu_i^*}{\sigma_*}\right)\right]}$$

$$= h(\varepsilon_i) (\text{say}) \Rightarrow \varepsilon_i = h^{-1}(\hat{u}_i) = g(\hat{u}_i) (\text{say}),$$

where  $\phi(\cdot)$  is the probability density function of a standard normal variable,  $f_u(u_i | \varepsilon_i) = f_{u,\varepsilon}(u_i, \varepsilon_i) / f_\varepsilon(\varepsilon_i)$  and:

$$\hat{u}_i = h(\varepsilon_i) = \mu_i^* + \sigma_* \frac{\phi\left(-\frac{\mu_i^*}{\sigma_*}\right)}{\left[1 - \Phi\left(-\frac{\mu_i^*}{\sigma_*}\right)\right]} = \mu_i^* + \sigma_* \lambda\left(-\frac{\mu_i^*}{\sigma_*}\right) \quad \text{where } \lambda(s) = \frac{\phi(s)}{[1 - \Phi(s)]}.$$

Hence:

$$\frac{\partial \hat{u}_i}{\partial \varepsilon_i} = -\frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2} + \sigma_* \times \lambda' \left( -\frac{\mu_i^*}{\sigma_*} \right) \times \frac{\partial}{\partial \varepsilon_i} \left( -\frac{\mu_i^*}{\sigma_*} \right) = \gamma \left[ \lambda' \left( -\frac{\mu_i^*}{\sigma_*} \right) - 1 \right],$$

where  $\lambda'(s) = -s\lambda(s) + \lambda^2(s)$ . Therefore,

(ix)  $f_{\hat{u}}(\hat{u}_i) = f_\varepsilon(g(\hat{u}_i)) \times |g'(\hat{u}_i)|$  (using (a))

$$= \frac{\exp \left[ -\frac{1}{2} \left( \frac{z_i \delta + \varepsilon_i}{\sqrt{\sigma_u^2 + \sigma_v^2}} \right)^2 \right] \Phi\left(\frac{\mu_i^*}{\sigma_*}\right)}{\sqrt{2\Pi(\sigma_u^2 + \sigma_v^2)} \Phi\left(\frac{z_i \delta}{\sigma_u}\right)} \times \gamma \left[ \frac{\left(\frac{\mu_i^*}{\sigma_*}\right) \left\{ \frac{\phi\left(\frac{\mu_i^*}{\sigma_*}\right)}{\Phi\left(\frac{\mu_i^*}{\sigma_*}\right)} \right\} + \left\{ \frac{\phi\left(\frac{\mu_i^*}{\sigma_*}\right)}{\Phi\left(\frac{\mu_i^*}{\sigma_*}\right)} \right\}^2 - 1 \right]^{-1}.$$

Imposing a half-normal restriction ( $\mu_i = z_i \delta = 0$ ) in (ix) we get:

$$(x) \quad f_{\hat{u}_i}(\hat{u}_i) = \frac{2 \exp \left[ -\frac{1}{2} \left( \frac{\varepsilon_i}{\sqrt{\sigma_u^2 + \sigma_v^2}} \right)^2 \right]}{\sqrt{2\Pi(\sigma_u^2 + \sigma_v^2)}} \times \Phi \left( \frac{-\varepsilon_i \sigma_u}{\sigma_v \sqrt{\sigma_u^2 + \sigma_v^2}} \right) \\ \times \left[ \gamma \left[ \left( \frac{-\varepsilon_i \sigma_u}{\sigma_v \sqrt{\sigma_u^2 + \sigma_v^2}} \right) \left\{ \frac{\phi \left( \frac{-\varepsilon_i \sigma_u}{\sigma_v \sqrt{\sigma_u^2 + \sigma_v^2}} \right)}{\Phi \left( \frac{-\varepsilon_i \sigma_u}{\sigma_v \sqrt{\sigma_u^2 + \sigma_v^2}} \right)} \right\} + \left\{ \frac{\phi \left( \frac{-\varepsilon_i \sigma_u}{\sigma_v \sqrt{\sigma_u^2 + \sigma_v^2}} \right)}{\Phi \left( \frac{-\varepsilon_i \sigma_u}{\sigma_v \sqrt{\sigma_u^2 + \sigma_v^2}} \right)} \right\}^2 - 1 \right] \right]^{-1}.$$

Now it is straightforward to see that (x) will coincide with expression (8) of Wang and Schmidt (2009) if we incorporate their notations  $a = \sqrt{\sigma_u^2 + \sigma_v^2}$ ,  $b = \sigma_u/\sigma_v$ , and  $\sigma_0 = (\sigma_v/\sigma_u)\sqrt{\sigma_u^2 + \sigma_v^2}$  into (x) and  $\varepsilon_i$  is replaced by  $g(\hat{u}_i)$  (as denoted above). Also:

$$(xi) \quad E(u_i) = \frac{1}{\sigma_u \sqrt{2\Pi}} \cdot \frac{1}{\Phi \left( \frac{z_i \delta}{\sigma_u} \right)} \int_0^\infty \exp \left[ -\frac{1}{2} \left( \frac{u_i - z_i \delta}{\sigma_u} \right)^2 \right] = z_i \delta + \sigma_u \times \lambda \left( -\frac{z_i \delta}{\sigma_u} \right).$$

We want to mention here some facts about the inverse Mill's ratio  $\lambda(s) = \phi(s)/[1 - \Phi(s)]$  stated above. It is straightforward that as  $s \rightarrow -\infty$ , (i)  $\lambda(s) \rightarrow 0$ , (ii)  $s\lambda(s) \rightarrow 0$ , and (iii)  $\lambda'(s) \rightarrow 0$  (see also Wang and Schmidt, 2009).

**Theorem 1a.** As  $\sigma_v^2 \rightarrow 0$  (with fixed  $\sigma_u^2$ ),  $\hat{u}_i - u_i \rightarrow_p 0 \quad \forall i$ .

**Proof.** Consider the expression of  $\hat{u}_i$  in (viii), i.e.,  $\hat{u}_i = \mu_i^* + \sigma_* \lambda(-\mu_i^*/\sigma_*)$ . As  $\sigma_v^2 \rightarrow 0 \Rightarrow \mu_i^* \rightarrow -\varepsilon_i \rightarrow_p u_i$  since  $\sigma_v^2 \rightarrow 0 \Rightarrow v_i \rightarrow_p 0 \quad \forall i$ . Again,  $\sigma_v^2 \rightarrow 0 \Rightarrow \sigma_* \rightarrow 0 \Rightarrow \sigma_* \lambda(-\mu_i^*/\sigma_*) \rightarrow 0 \times \lambda(-\infty) \rightarrow 0$ . Therefore,  $\sigma_v^2 \rightarrow 0 \Rightarrow (\hat{u}_i - u_i) \rightarrow_p 0$ .

**Theorem 1b.** As  $\sigma_v^2 \rightarrow \infty$  (with fixed  $\sigma_u^2$ ),  $\hat{u}_i \rightarrow_p E(u_i) \quad \forall i$ .

**Proof.** Consider the  $\hat{u}_i = \mu_i^* + \sigma_* \lambda(-\mu_i^*/\sigma_*)$  in (viii). As  $\sigma_v^2 \rightarrow \infty \Rightarrow \mu_i^* \rightarrow z_i \delta$ . Again,  $\sigma_v^2 \rightarrow \infty \Rightarrow \sigma_* \rightarrow \sigma_u \Rightarrow \sigma_* \lambda(-\mu_i^*/\sigma_*) \rightarrow \sigma_u \times \lambda(-z_i \delta/\sigma_u)$ . Therefore,  $\sigma_v^2 \rightarrow \infty \Rightarrow \hat{u}_i \rightarrow_p \{z_i \delta + \sigma_u \times \lambda(-z_i \delta/\sigma_u)\} = E(u_i)$  using (xi).

**Theorem 1c.** As  $\sigma_v^2 \rightarrow 0$  (with fixed  $\sigma_u^2$ ),  $f_{\hat{u}_i} \rightarrow f_{u_i} \quad \forall i$ .

**Proof.** Consider the expression of  $f_{\hat{u}_i}$  in (ix):

$$f_{\hat{u}_i} = \frac{\exp \left[ -\frac{1}{2} \left( \frac{z_i \delta + \varepsilon_i}{\sqrt{\sigma_u^2 + \sigma_v^2}} \right)^2 \right]}{\sqrt{2\Pi(\sigma_u^2 + \sigma_v^2)}} \times \frac{\Phi \left( \frac{\mu_i^*}{\sigma_*} \right)}{\Phi \left( \frac{z_i \delta}{\sigma_u} \right)} \times |g'(\hat{u}_i)|.$$

Since  $\partial \hat{u}_i / \partial \varepsilon_i = \gamma[\lambda'(-\mu_i^*/\sigma_*) - 1]$  and  $\sigma_v^2 \rightarrow 0 \Rightarrow \sigma_* \rightarrow 0$ , the Jacobian will be 1. As  $\sigma_v^2 \rightarrow 0$ , the rest of the expression for  $f_{\hat{u}_i}$  will tend to:

$$\frac{1}{\sigma_u \sqrt{2\Pi}} \exp\left[-\frac{1}{2}\left(\frac{u_i - z_i \delta}{\sigma_u}\right)^2\right] \Phi\left(\frac{z_i \delta}{\sigma_u}\right),$$

which is the distribution assumed for the inefficiency error term  $u_i$ .

**Theorem 1d.** As  $\sigma_v^2 \rightarrow \infty$  (with fixed  $\sigma_u^2$ ),  $(\Pi/\Pi - 2)(\sigma_v/\sigma_u^2)[\hat{u}_i - E(u_i)] \rightarrow_d N(0,1) \forall i$ .

**Proof.** Consider the expression  $(\sigma_v/\sigma_u^2)[\hat{u}_i - E(u_i)]$ . This can be written:

$$-\frac{\sigma_v}{\sigma_u^2} \times \gamma \times \varepsilon_i + \frac{\sigma_v}{\sigma_u^2} \left[ -\gamma(z_i \delta) + \sigma_* \lambda\left(-\frac{\mu_i^*}{\sigma_*}\right) - \sigma_u \lambda\left(-\frac{(z_i \delta)}{\sigma_u}\right) \right].$$

The first term is:

$$\frac{\sigma_v}{\sigma_u^2 + \sigma_v^2} \times u_i - \frac{\sigma_v^2}{\sigma_u^2 + \sigma_v^2} \times \frac{v_i}{\sigma_v} \cong -\frac{v_i}{\sigma_v},$$

which follows a standard normal distribution since  $[\sigma_v/(\sigma_u^2 + \sigma_v^2)] \rightarrow 0$  and  $[\sigma_v^2/(\sigma_u^2 + \sigma_v^2)] \rightarrow 1$  as  $\sigma_v^2 \rightarrow \infty$ . The second term is:

$$\begin{aligned} & \frac{\sigma_v}{\sigma_u^2} \left[ -\gamma(z_i \delta) + \sigma_* \lambda\left(-\frac{\mu_i^*}{\sigma_*}\right) - \sigma_u \lambda\left(-\frac{(z_i \delta)}{\sigma_u}\right) \right] \\ & \cong \frac{\sigma_v}{\sigma_u^2} \left[ -\gamma(z_i \delta) + \sigma_* \sqrt{\frac{2}{\Pi}} - \frac{2}{\Pi} \mu_i^* - \sigma_u \sqrt{\frac{2}{\Pi}} + \frac{2}{\Pi} (z_i \delta) \right], \end{aligned}$$

using the mean value theorem, i.e.,  $\lambda(k) \cong \lambda(0) + \lambda'(0) \times k$ , where  $\lambda(0) = \sqrt{2/\Pi}$  and hence  $\lambda'(0) = -0 \times \lambda(0) + \lambda^2(0) = 2/\Pi$ . Thus the second term:  $\cong (2/\Pi) \times (v_i/\sigma_v)$  as  $\sigma_v^2 \rightarrow \infty$ . Therefore,

$$\left(\frac{\sigma_v}{\sigma_u^2}\right)[\hat{u}_i - E(u_i)] = \left\{ \left(-1 + \frac{2}{\Pi}\right) \times \frac{v_i}{\sigma_v} \right\} \sim N\left(0, \left(\frac{\Pi - 2}{\Pi}\right)^2\right), \text{ which proves 1d.}$$

Thus it is easy to see that the estimated inefficiency error under the general assumptions we have considered has similar statistical properties if one assumes a half normal distribution (see Theorem 1 in Wang and Schmidt, 2009).

We also want to show the probability density function of the estimated level of technical efficiency. Using the Jondrow et al. (1982) technique, expression (i), (v) and (viii), we get that the estimated TE is:

$$TE_i = E(\exp(-u_i) | \varepsilon_i) = \int_0^{\infty} \exp(-u_i) f_u(u_i | \varepsilon_i) du_i$$

$$= \exp\left[-\mu_i^* + \frac{\sigma_*^2}{2}\right] \left\{ \frac{\Phi\left(\frac{\mu_i^* - \sigma_*}{\sigma_*}\right)}{\Phi\left(\frac{\mu_i^*}{\sigma_*}\right)} \right\}$$

$$= j(\varepsilon_i)(\text{say}) \Rightarrow \varepsilon_i = j^{-1}(TE_i) = k(TE_i)(\text{say}).$$

Hence,

$$\frac{\partial}{\partial \varepsilon_i} (\ln TE_i) = \frac{\gamma}{\sigma_*} \left[ \sigma_* - \frac{\phi\left(\frac{\mu_i^* - \sigma_*}{\sigma_*}\right)}{\Phi\left(\frac{\mu_i^* - \sigma_*}{\sigma_*}\right)} + \frac{\phi\left(\frac{\mu_i^*}{\sigma_*}\right)}{\Phi\left(\frac{\mu_i^*}{\sigma_*}\right)} \right].$$

Therefore,

$$\frac{\partial TE_i}{\partial \varepsilon_i} = TE_i \times \frac{\partial}{\partial \varepsilon_i} (\ln TE_i)$$

$$= \exp\left[-\mu_i^* + \frac{\sigma_*^2}{2}\right] \left\{ \frac{\Phi\left(\frac{\mu_i^* - \sigma_*}{\sigma_*}\right)}{\Phi\left(\frac{\mu_i^*}{\sigma_*}\right)} \right\} \left[ \frac{\gamma}{\sigma_*} \left[ \sigma_* - \frac{\phi\left(\frac{\mu_i^* - \sigma_*}{\sigma_*}\right)}{\Phi\left(\frac{\mu_i^* - \sigma_*}{\sigma_*}\right)} + \frac{\phi\left(\frac{\mu_i^*}{\sigma_*}\right)}{\Phi\left(\frac{\mu_i^*}{\sigma_*}\right)} \right] \right].$$

Now, using the result (a) stated above we can write:

$$(xii) \quad f_{TE}(TE_i) = f_\varepsilon(\varepsilon_i) \times \left| \frac{\partial \varepsilon_i}{\partial TE_i} \right| = f_\varepsilon(\varepsilon_i) \times \left| \frac{\partial TE_i}{\partial \varepsilon_i} \right|$$

$$= \frac{\sigma_v}{\sigma_u \sqrt{2\pi}} \times \frac{\Phi^2\left(\frac{\mu_i^*}{\sigma_*}\right)}{\Phi\left(\frac{z_i \delta}{\sigma_u}\right) \Phi\left(\frac{\mu_i^* - \sigma_*}{\sigma_*}\right)} \times \exp\left[ \mu_i^* - \frac{\sigma_*^2}{2} - \frac{1}{2} \left\{ \left( \frac{z_i \delta + \varepsilon_i}{\sqrt{\sigma_u^2 + \sigma_v^2}} \right)^2 \right\} \right]$$

$$\times \left[ \sigma_* - \left\{ \frac{\phi\left(\frac{\mu_i^* - \sigma_*}{\sigma_*}\right)}{\Phi\left(\frac{\mu_i^* - \sigma_*}{\sigma_*}\right)} \right\} + \left\{ \frac{\phi\left(\frac{\mu_i^*}{\sigma_*}\right)}{\Phi\left(\frac{\mu_i^*}{\sigma_*}\right)} \right\} \right]^{-1}.$$

### 3. Empirical Model Considered

We considered the famous translog form of the production function to explain the  $i$  th firm's production behavior given three input variables—the value of intermediate inputs ( $I_i$ ), the amount of labor ( $L_i$ ), and the value of fixed capital ( $FA_i$ ). The frontier function has the following form:

$$\begin{aligned} \ln Y_i = & \beta_0 + \beta_1 \ln I_i + \beta_2 \ln L_i + \beta_3 \ln FA_i + \beta_{11} (\ln I_i)^2 + \beta_{22} (\ln L_i)^2 \\ & + \beta_{33} (\ln FA_i)^2 + \beta_{12} (\ln I_i)(\ln L_i) + \beta_{13} (\ln I_i)(\ln FA_i) \\ & + \beta_{23} (\ln L_i)(\ln FA_i) + v_i - u_i, \end{aligned} \quad (1)$$

where  $\beta_j$  is the coefficient of the covariate  $\ln X_{ji}$ , where  $X_{ji}$  is the amount of the  $j$ th input ( $j = 1, 2, 3$ ) used by the  $i$ th firm,  $\beta_0$  is the constant term, and the other coefficients  $\beta_{ji}$ 's are taken to be symmetric. The usual two-sided statistical noise  $v_i$  and the one-sided nonnegative inefficiency error  $u_i$  are assumed to be distributed independently. As in Lundvall and Battese (2000), Bhandari and Maiti (2007), and many others, the  $i$ th firm's inefficiency term is taken to depend on firm-specific variables like size of the firm (measured by the intermediate inputs used,  $I_i$ ), its age ( $Age_i$ ), and so on, and their associated parameters (denoted  $\delta$ ) along with those of the SFPF will be estimated through a single-stage maximum likelihood estimation (MLE) method (see for details Battese et al., 1988, 1993). This also allows for the assumption that the inputs used in the deterministic part of the stochastic frontier,  $X$ , and the variables used to explain the inefficiency term may overlap. In other words, the position of the frontier may depend on things other than inputs while some of the inputs themselves may also affect TE (Wang and Schmidt, 2002). In addition, we assume that the two-sided random error term is distributed independently of the regressors and of the inefficiency-explaining variables. Thus the following inefficiency sub-model is added to the SFPF model in (1):

$$\begin{aligned} \mu_i = & \delta_0 + \delta_1 \ln(I_i) + \delta_2 \ln(Age_i) + \delta_{11} \{\ln(I_i)\}^2 + \delta_{22} \{\ln(Age_i)\}^2 \\ & + \delta_{12} \{\ln(I_i)\} \{\ln(Age_i)\} + \delta_{01} SD_1 + \delta_{02} SD_2 + \delta_{03} OD, \end{aligned} \quad (2)$$

where  $SD_1$ ,  $SD_2$ , and  $OD$  are the three different (intercept) dummy variables. The first two are used to distinguish firms located in two different groups of Indian states while the last one is introduced to form groups of firms belonging to two alternative types of organizations.

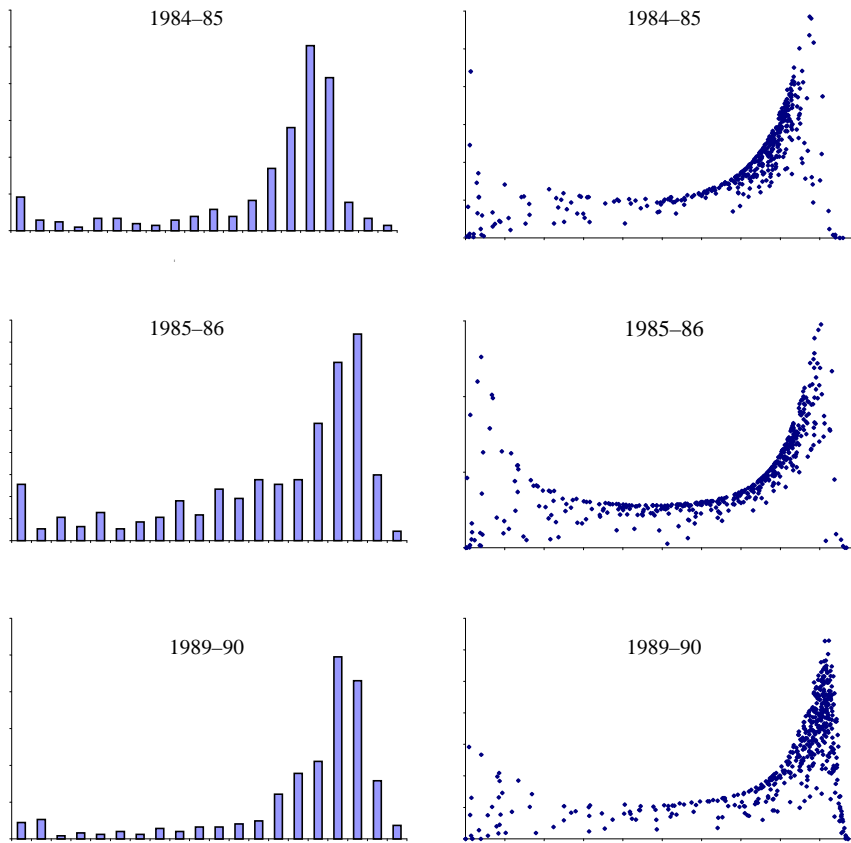
To estimate model (1)–(2) we use cross-sectional micro-level data on the Indian leather industry collected by the Central Statistical Organization (CSO), Government of India through its Annual Survey of Industries (ASI) and made available electronically for years 1984–85, 1985–86, 1989–90, 1990–91, 1994–95, 1999–2000, and 2002–03. Our data cover the entire organized leather sector, i.e., the part of the industry for which ASI data are published by CSO on a regular basis. Although an overwhelming majority of the production of Indian leather industry comes from its un-organized segment,<sup>2</sup> it is very difficult to do similar kind of analysis for this segment due to non-availability of reliable data.

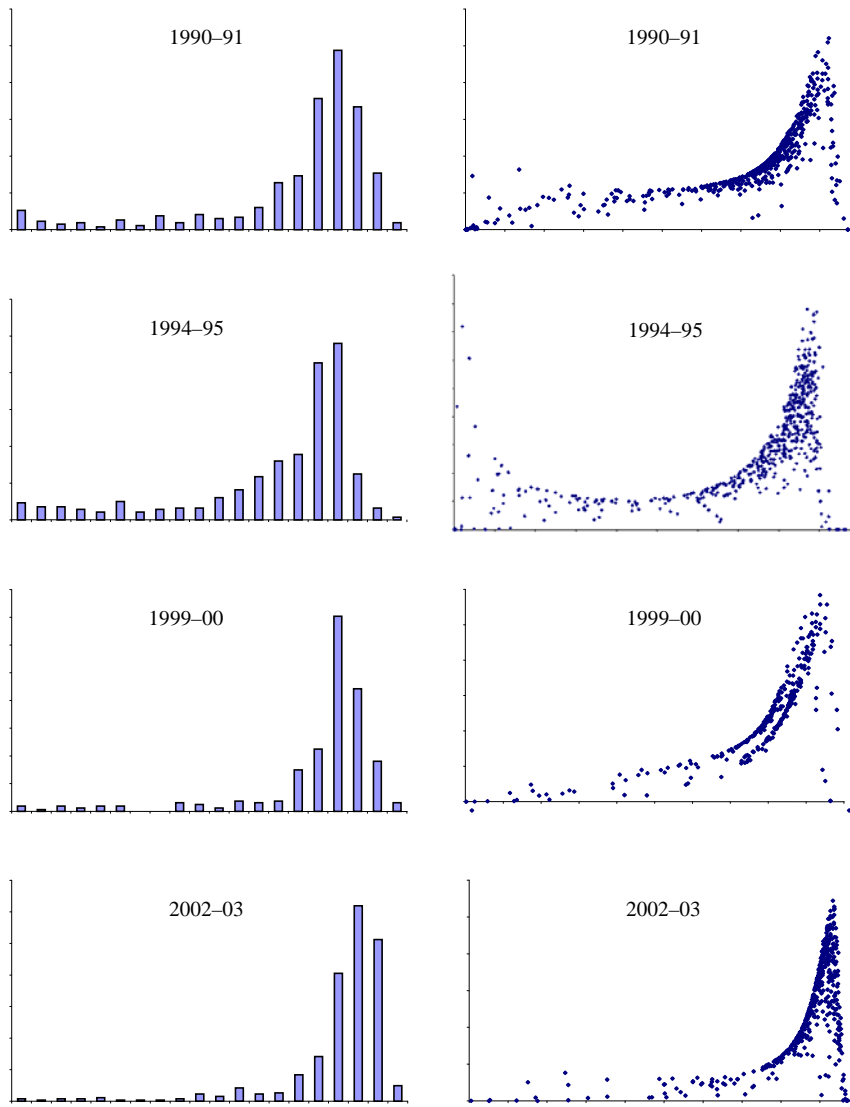
Using the software package FRONTIER 4.1 we estimated the model and obtained the TE score of each firm for each year. We plotted proportions of firms (on the vertical axis) over intervals 0.00–0.05, 0.05–0.10, ..., 0.95–1.00 of the obtained TE scores to construct histograms for each year to show the frequency distribution of the estimated TE scores. Detailed results including these histograms are to be published elsewhere (Bhandari and Maiti, forthcoming). However, we



reproduce the histograms in the left panel of the Figure 1 here to facilitate comparison with the theoretically derived probability density of the estimated TE, which is shown in the right panel. The histograms show a very high negatively skewed distribution for each year. In other words, an overwhelming majority of the firms have a very high level of TE and most of the remaining firms are distributed to the left of the mode of the distribution to form a very long negative tail.

**Figure 1: Histograms (Left) Showing Proportions of Firms (Vertical Axis) over 20 Equal-Width Intervals of TE Scores (Horizontal Axis) and the Probability Density (Right) of Estimated TE (Vertical Axis) against Estimated TE (Horizontal Axis)**



**Figure 1 (Continued)**

#### **4. Conclusion**

The stochastic frontier production function model has been widely used since its inception in 1977 to estimate technical efficiency of production units. It is also

not uncommon in the literature to empirically estimate such technical efficiencies using readily available statistical software packages, e.g., LIMDEP, FRONTIER, STATA, and to study the distribution of estimated efficiencies. However, no serious attempt has been made to find the distribution of the estimated one-sided inefficiency error or the estimated efficiency levels. It is to be noted in this context that the distribution of the estimated one-sided error may not be identical to the one which is theoretically assumed for the population. Only recently Wang and Schmidt (2009) theoretically derived the distribution of the estimated one-sided inefficiency error term assuming that it is a half normal distribution. This paper attempts to generalize the Wang and Schmidt (2009) results by assuming that the inefficiency error has a general truncated normal, instead of the restrictive half normal, distribution. We also state and prove several important statistical properties. In this connection we also derive the theoretical probability density of the estimated TE. We also compare the observed distributions of TE with theoretical counterparts. It is clear from Figure 1 that these two sets of distributions are at least visually consistent with each other. However, alternative goodness-of-fit statistics (Kolmogorov-Smirnov or Personian  $\chi^2$  or both) can be used to test whether the discrepancy between these two sets of distributions is statistically significant using methods in Wang et al. (2011).

### Notes

1. Kopp et al. (1990) uses a generalized method of moments estimation procedure when  $v_i$  is not normal.
2. For instance, according to the latest data available at the CSO, the share of unregistered manufacturing sector of the leather and fur products industry was more than 77% in 2004–05. Although this has come down gradually, it was still as much as 64% in 2008–09.

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