

A Procedure for Testing Granger Causality of Infinite Order

Fathali Firoozi and Donald Lien*

Department of Economics, University of Texas at San Antonio, U.S.A.

Key words: autoregressive; infinite lags; dynamic models

JEL classification: C12; C32

1. Introduction

Let $\{X_t\}$ and $\{Y_t\}$ be two single-variable stationary time series. X is said to fail Granger-cause Y if the following equality of conditional expectations holds (Granger, 1969):

$$E(Y_t | Y_{t-1}, X_{t-1}, X_{t-2}, \dots) = E(Y_t | Y_{t-1}). \quad (1)$$

Checking for Granger causality has now become a standard procedure in applied time series (Enders, 2010; Greene, 2008; Lutkepohl, 2006). The conventional testing procedure formulates the problem in a vector autoregressive (VAR) format that summarizes the following system of two regressions:

$$Y_t = \mu_1 + \gamma_1 Y_{t-1} + \alpha_1 X_{t-1} + \varepsilon_{1t}, \quad (2)$$

$$X_t = \mu_2 + \delta_1 X_{t-1} + \beta_1 Y_{t-1} + \varepsilon_{2t}. \quad (3)$$

The conventional procedure for testing the hypothesis that X fails to Granger cause Y tests the null hypothesis:

$$H_0 : \alpha_1 = 0 \quad (4)$$

on the system in (2)-(3). The test is the usual χ^2 or F -test (in the multivariable case) or the z -test or t -test (in the single variable case). The disturbances ε_1 and ε_2 are assumed to have the classical structure, e.g., each satisfies the assumptions of homoscedasticity and no-autocorrelation, but there is cross-equation correlation so that the covariance matrix for the system is defined by:

*Correspondence to: Department of Economics, University of Texas at San Antonio, San Antonio, TX 78249, U.S.A. E-mail: Don.Lien@utsa.edu. The authors wish to thank an anonymous referee for helpful and constructive comments. Fathali Firoozi acknowledges summer support from COB at UTSA.

$$\Omega = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \otimes I = \Sigma \otimes I, \quad (5)$$

where, for a sample of size T , I is the $(T \times T)$ identity matrix. A feature of the system in (2)-(3) is that the regressors in the two equations are identical and, hence, ordinary least squares (OLS) and generalized least squares (GLS) generate the same parameter estimates. However, the efficiency of the estimated variances for the parameters requires a consistent estimate of $\Sigma = [\sigma_{ij}]$. Since the single-equation OLS estimates are consistent, the corresponding residuals for the two equations are then used to produce consistent estimates for σ_{ij} . The procedure generates a consistent estimate $\hat{\Omega}$ of Ω , which is utilized in the stated test procedure.

A shortcoming of the conventional Granger causality test as summarized above is that it actually tests the following limited version of the non-causality statement in (1):

$$E(Y_t | Y_{t-1}, X_{t-1}) = E(Y_t | Y_{t-1}), \quad (6)$$

which is the Granger causality of order one. As noted above, a number of theoretical and computational simplifications emerge as a consequence of the stated limitation to one lag of X . A more general version of the non-causality statement is defined for a positive integer p as:

$$E(Y_t | Y_{t-1}, X_{t-1}, X_{t-2}, \dots, X_{t-p}) = E(Y_t | Y_{t-1}), \quad (7)$$

This is the order- p statement of the causality and can be tested by extending the system in (2)-(3) to:

$$Y_t = \mu_1 + \gamma_1 Y_{t-1} + \alpha_1 X_{t-1} + \dots + \alpha_p X_{t-p} + \varepsilon_{1t}, \quad (8)$$

$$X_t = \mu_2 + \delta_1 X_{t-1} + \beta_1 Y_{t-1} + \dots + \beta_p Y_{t-p} + \varepsilon_{2t}. \quad (9)$$

The non-causality test is then a test of the joint hypothesis:

$$H'_0 : \alpha_1 = \alpha_2 = \dots = \alpha_p = 0 \quad (10)$$

on model (8)-(9), which is again a conventional χ^2 -test (for large T) or F -test (for moderate T) on model (8)-(9). The computational simplification in model (2)-(3) reflected by having the same regressors in the two equations has now disappeared in model (8)-(9) for $p \geq 2$. Hence, for $p \geq 2$, the single-equation OLS, although still consistent, is inefficient for estimation of the system in (8)-(9) and a more general version of the seemingly unrelated regressions method is used for estimation of the parameters and the system covariance matrix. Also, for $p \geq 2$, a loss in degree of freedom by $(p-1)$ emerges as one switches from the order one system in (2)-(3) to the larger p -order system in (8)-(9). For the stated finite-order

cases, this relative loss of $p-1$ in degrees of freedom is clearly an increasing function of the lag order, p .

The main challenge in the stated finite-order cases is the determination of the lag order, p . It is well known that the appropriate refinement for determining the lag order is from general to specific, that is, start with the largest lag order and then test the restrictions that generate smaller orders. This is mainly because of the omitted variable bias that is associated with the reverse refining from a smaller lag order to a larger lag order. This refinement principle provides one compelling reason to start from infinite lags. There are two other fundamental reasons for working with causality of infinite order. First, the Granger causality in its most general form contains infinite lags as reflected in (1) above. Second, as will be shown, the use of infinite lags under a rational structure allows reducing the relative loss of degrees of freedom associated with the number of lags from $p-1$ for the order- p system in (8)-(9) to only one for the infinite-order case.

2. Testing Causality of Infinite Order

The last section summarized the foundations of the existing finite-order Granger causality tests and pointed out three of the theoretical and applied reasons for using a test of infinite-order causality in conjunction with the conventional tests of finite-order causality. The stated three reasons are associated with the well known refinement principle for determining the lag order, the general statement of Granger causality, and the loss of degrees of freedom in the finite-order cases. In this section we propose a simple procedure for testing infinite-order Granger causality. The literature has shown various complications that arise in contexts that incorporate infinite lags. For instance, the core studies of Saikkonen and Lutkepohl (1996) and Lutkepohl and Saikkonen (1997) rigorously highlighted some of the complications and the needed finite-order approximations in the general area of VAR analysis with infinite lags. Our focus here is on testing infinite-order causality in a simple construction that is free of any finite-order approximation and relies only on a mainstream assumption.

The second equation in both of the finite-order systems in (2)-(3) and (8)-(9) are mirror images of the first equation in the sense that the second equation involves only switching of the two variables and introduction of new parameters. Accordingly, we present construction of the first equation for the case of infinite lags and then generate the second equation of its system. The generalization of (8) to infinite lags is:

$$Y_t = \mu_1 + \gamma_1 Y_{t-1} + \sum_{j=1}^{\infty} \alpha_j X_{t-j} + \varepsilon_{1t} . \quad (11)$$

In a study with an objective similar to the one in the present study, Lutkepohl and Poskitt (1996) considered a multivariate version of (11) and assumed a finite-order lag (h) for variable X , instead of infinite order in our study here. The

infinite-order feature of Lutkepohl and Poskitt (1996) emerges from a central assumption (Assumption 1, p. 66) that defines the lag order h as a specific function of the sample size (T) so that h converges to infinity as T converges to infinity with a specific finite sum property. Lutkepohl and Poskitt (1996) then present an asymptotic Wald (χ^2 -squared) test of Granger causality and then appeal to the stated assumption to argue that the test is asymptotically valid in the infinite-order case. The approach in our study here is fundamentally different. We employ the geometric structure for the infinite lag as an alternative approach and present a test that has well known properties.

We now adopt a popular structure for the lag effect, namely, the geometric structure. The structure states that the effect of the lag sequence decreases over time and this decrease has a geometric pattern. Specifically, we assume

$$\alpha_j = \pi\lambda^j, \text{ for some } \pi \text{ and } \lambda \text{ with } |\lambda| < 1. \quad (12)$$

Therefore, with a use of this assumption, the lag operator (L), and the geometric series, (11) reduces to:

$$\begin{aligned} Y_t &= \mu_1 + \gamma_1 Y_{t-1} + \pi \sum_{j=0}^{\infty} (\lambda L)^j X_t - \pi X_t + \varepsilon_{1t} \\ &= \mu_1 + \gamma_1 Y_{t-1} + \frac{\pi\lambda L}{(1-\lambda L)} X_t + \varepsilon_{1t}. \end{aligned} \quad (13)$$

Multiplying both sides by $(1-\lambda L)$ reduces (13) to:

$$(1-\lambda L)Y_t = (1-\lambda)\mu_1 + \gamma_1(1-\lambda L)Y_{t-1} + \pi\lambda L X_t + (1-\lambda L)\varepsilon_{1t}, \quad (14)$$

which after rearrangements reduces to:

$$Y_t = (1-\lambda)\mu_1 + (\lambda + \gamma_1)Y_{t-1} - \gamma_1\lambda Y_{t-2} + \pi\lambda X_{t-1} + \eta_{1t}, \quad (15)$$

where $\eta_{1t} = (1-\lambda L)\varepsilon_{1t}$. Equation (15) is an infinite-order analog of the first equation in system (8)-(9). Also, it follows from the assumption in (12) that the infinite-order analog of the finite-order non-causality hypothesis H'_0 in (10) above is now:

$$H''_0: \pi\lambda = 0. \quad (16)$$

The infinite-order causality test is equivalent to testing H''_0 on a two-equation system that consists of (15) and its counterpart for X_t . However, there are some complications that need to be addressed. As shown by (15), the two regressions in the corresponding system do not have identical regressors. Further, the parametric restriction in (16), the regression in (15), and its counterpart regression for X_t are all nonlinear. Hence, it appears that one needs a nonlinear system method for estimation of the system and a covariance approximation to test the nonlinear

restriction in (16). But, for the purpose of causality testing, the relevant structure of the problem turns out to be completely linear. To show this, define:

$$\varphi_0 = (1 - \lambda)\mu_1, \quad (17)$$

$$\varphi_1 = (\lambda + \gamma_1), \quad (18)$$

$$\varphi_2 = -\gamma_1\lambda, \quad (19)$$

$$\varphi_3 = \pi\lambda. \quad (20)$$

Then the regression in (15) is written equivalently as

$$Y_t = \varphi_0 + \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + \varphi_3 X_{t-1} + \eta_{1t}. \quad (21)$$

It is also clear from the definitions in (17)-(20) that the nonlinear restriction $\pi\lambda = 0$ in (16) on the nonlinear model in (15) is equivalent to the linear restriction $\varphi_3 = 0$ on the linear model in (21). Thus we have the following result.

Theorem 1. Testing the infinite-order Granger non-causality statement in (1) under the geometric lag structure is equivalent to testing the simple linear hypothesis:

$$H_0''' : \varphi_3 = 0 \quad (22)$$

on the linear system

$$Y_t = \varphi_0 + \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + \varphi_3 X_{t-1} + \eta_{1t}, \quad (23)$$

$$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 Y_{t-1} + \eta_{2t}. \quad (24)$$

The system in (23)-(24) is estimated by the methods discussed in Greene (2008, pp. 651-652) and Hatanaka (1974, 1976), and then the standard χ^2 -test (multivariable, large T), F -test (multivariable, moderate T), t -test (single variable, moderate T), or z -test (single variable, large T) is used to test the hypothesis in (22) on this system. It is clear that the loss of degrees of freedom in terms of lags in the infinite-order model of (23)-(24) is only one relative to the order one model in (2)-(3), which is a smaller loss than the relative loss of $p-1$ associated with the order p model in (8)-(9) when $p \geq 3$.

Some of the distinguishing features of the approach here can now be highlighted. The ARMA(p, q) model uses the finite-order assumption to approximate the infinite order. Some of the shortcomings of the finite-order assumption were highlighted in Section 1. Alternatively, (12) allows maintaining the infinite order but imposes the geometric structure. A similar approach is adopted in Engle (1982) where linear decreasing weights analogous to the geometric lag structure are imposed on the past innovations to describe the time-varying conditional variance. Note that a reduced form of (12) constitutes a special case of ARMA(p, q) with parameter restrictions.

Infinite-order lag models in macroeconomics time series also arise as a consequence of applying a popular mechanism of expectation formation to a series.

A typical case is presented in Greene (2008, pp. 678-680) in which an application of the mechanism to inflation rate leads to an infinite-order distributed lag model where the coefficients satisfy the geometric structure. While theoretically well grounded, the geometric structure may be a misspecification. Aaker et al. (1982) provide an empirical example. They suggest use of cross-correlation and residual autocorrelation to detect possible misspecification.

Adding more lagged orders on Y into the information set in (6) and (7) does not affect the analysis here since the test focus is on the coefficients of X . Specifically, it will lead to higher-order lagged terms for Y in (15). The non-causality test procedure remains the same. However, empirically, more lagged Y terms introduce more nuisance parameters and reduce the estimation precision, thus the test results become less conclusive. An initial lag order selection for the response variable Y may be applied using some of the available criteria in the literature (Lutkepohl and Poskitt, 1996, p. 82; Lutkepohl, 2006, chapter 4; Enders, 2010, p. 70).

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