

Portfolio Selections with Innate Learning Ability

Jin-Ray Lu*

Department of Finance, National Dong Hwa University, Taiwan

Chih-Ming Chan

Department of Finance, National Chung Cheng University, Taiwan

Wen-Shen Li

Department of Finance, National Dong Hwa University, Taiwan

Abstract

This study explores how innate learning ability changes portfolio selection decision-making in a continuous-time framework. We re-solve Samuelson-Merton's portfolio choice problem framed in a fixed investment opportunity set for an individual with a learning ability. In contrast to traditional theoretical results, we suggest that risk-averse investors with a risk-cognitive ability hold a lower fraction of risky stocks to hedge against the jump risk and volatility risk since the investors are cognizant of the market risks. In addition, an individual whose learning process correlates strongly with stock movements would be likely to invest more in stocks.

Key words: learning; asset allocation; cognition

JEL classification: G11; C61

1. Introduction

Asset allocation decision-making is a central issue in modern financial theory. Samuelson (1969) and Merton (1971, 1969) first proposed a continuous-time decision model of portfolio selections and consumption rules for a representative individual using a stochastic dynamic programming methodology. Many studies have analyzed how an individual allocates wealth capital between assets with various risk-return features subject to wealth constraints in a stochastic environment setting (see Basak and Chabakauri, 2010; Chellathurai and Draviam, 2007). In this study, we discuss how an individual's observations of market conditions affect his cognition of market risks, and how this learning process affects his optimal decisions regarding his portfolio selections. That is to say, this study analyzes *how* an individual's innate learning ability changes asset allocations.

*Correspondence to: Department of Finance, National Dong Hwa University, Section 2, 1 University Rd. Shou-Feng, Hualien 974, Taiwan, R.O.C. E-mail: jinray@mail.ndhu.edu.tw.

Many human activities have inherent learning characteristics whereby individuals can learn through observation, experience, participation, or education and training, thereby accumulating knowledge and techniques for increasing their decision-making ability and improving their performance. The various processes and effects of learning ability have been extensively discussed in the science of psychology; some studies also have analyzed learning process from the perspective of economics (see Mitropoulos, 2004; Börgers, 1996; Börgers and Sarin, 1997; Bikhchandani et al., 1998). The learning process is one of the rational behaviors of individuals (Simon, 1956). Many researchers have concerned themselves with how individuals make certain decisions through a learning process based upon their cognitions, experiences, or their observations of external information, in pursuit of their optimal objectives. It can be said that all voluntary human behavior or decision-making is based on learning, regardless of whether learning is accomplished through the accumulation of experience or simply by observing phenomena.

Nowadays, numerous financial studies that involve the learning process based on the concept of an estimation of parameter uncertainties have been carried out (see Bawa et al., 1979; Dothan and Feldman, 1986; Detemple, 1986; Brennan, 1998; Xia, 2001; Brennan and Xia, 2002; Brandt et al., 2005; Guidolin and Timmermann, 2007; Sharpe, 2010; Branch and Evans, 2010; Shiraishi, 2012). Such studies have estimated or predicted the uncertain parameters of stock returns or state variables for the purpose of precisely controlling and analyzing these parameters to assist in deciding upon optimal portfolio selections. In other words, individuals' adjusted asset allocations were calculated through optimal filtering learning equations to obtain mean values for the uncertain parameters of stock returns or state variables. The key results of these studies indicated that estimations, predictions, or learning about uncertain parameters were important for optimal asset allocation decision-making, especially for individuals with a long-term investment horizon.

In the original portfolio selection studies of Samuelson (1969) and Merton (1969, 1971), an investment opportunity set is fixed, in which asset parameters (i.e., a mean growth rate and a volatility rate of stock returns) and external parameters are time-invariant, given an investment horizon. Thus, it is possible to solve a fixed-mixed strategy for the asset allocations without the need to estimate, learn, or predict parameters. Furthermore, subsequent studies have augmented the Samuelson-Merton models by considering uncertain parameters. Brennan (1998), Xia (2001), and Brandt et al. (2005) proposed that individuals usually need to estimate these parameters, and this reliance on estimation impedes their making an optimal decision about asset allocations. Consequently, according to these previous studies, individuals need to estimate, forecast, learn, or judge these uncertain parameters in order to increase their control and decrease the level of uncertainty of these parameters, in order to arrive at an optimal decision in terms of asset allocations. That is, decision-makers can manage an estimation risk for these parameters through some learning mechanisms, which improves their ability to effectively allocate their assets.

The present study differs from numerous previous studies in that we focus on a purely innate learning ability rather than on an estimation or prediction of uncertain

parameters. We focus on how learning ability *itself* affects an individual's portfolio selections rather than on how decision-makers establish a learning process regarding uncertain parameters. This study discusses optimal portfolio selection decision-making for a representative individual who has an innate learning ability, and we analyze how the individuals' learning affects his asset allocations in a Samuelson-Merton environment where the parameters are certain. This learning ability is essential for making an optimal decision about asset allocations. That is to say, an individual with a strong learning ability regarding stock markets can more appropriately analyze, manage, and allocate his assets. Consequently, this study analyzes how the portfolio choices for individuals *with* an innate learning ability differ from the choices made by those *without* a learning ability, considering the problem within a Samuelson-Merton decision framework. Specifically, this study develops an innate learning process, in which we set out a stochastic dynamic process for describing an individual's learning process. Here, the stochastic learning process captures the individual's cognition of the risky assets' risks. As is well known, a true volatility rate of stock return cannot be precisely determined through the application of a stochastic volatility model, but given an appropriate description of stochastic volatility, the process can reasonably characterize the volatility dynamics, and we can describe a dynamic learning process by which the individual is cognizant of the varied circumstances of the financial markets.

Analyses by Brennan (1998), Xia (2001), and Cvitanić et al. (2006) are all closely related to our present study. These studies point out that individuals need to estimate uncertain parameters of stock returns or state variables under a time-varying investment opportunity set. That is to say, when individuals allocate their wealth across these assets, they need to first estimate or forecast numerous parameters of related variables, especially the mean returns of the stock. In contrast, the present study does not estimate or forecast uncertain stock parameters. In fact, this study focuses on the individual's purely innate learning process in terms of his cognition regarding financial markets. We simply examine *how* an individual with a learning ability makes decisions regarding asset allocations, where the related parameters of stock returns and state variables do not need to be estimated, since these parameters are taken to be known. Therefore, we have excluded the estimation risk of uncertain parameters proposed by Brennan (1998) and we analyze the individual's optimal portfolio selection decision-making under a constant investment opportunity set.¹

In short, in a rigid definition, the previous studies of portfolio selection which involve a learning mechanism all seem to take the form of statistical estimation investigations of uncertain parameters. These studies do not appear to investigate human being's inherent learning ability *per se*. Notably, previous research has been concerned solely with the estimation or prediction of uncertain parameters. However, our study specifically focuses on how learning *itself* affects asset allocations. To the best of our knowledge, there is no similar study which discusses the issue from both psychological and financial perspectives.

The study includes five parts. The next section reviews the literature on learning and asset allocations. Section 3 develops a continuous-time decision model for

analyzing optimal portfolio selection. Section 4 gives numerical examples for analyzing the sensitivities of holding weights in relation to learning effects. Section 5 concludes.

2. Literature Review

This section reviews the literature on learning, asset allocation, and the impacts of learning on portfolio selections.

2.1 Learning

Learning has often been discussed in both psychology and economics, and it is one of the main issues in psychological studies. Most economists, however, seldom define or explore learning's essence, choosing instead to apply the learning process to analyze human economic behavior (see Brenner, 1999). For example, through repeated trials, economists have tried to observe the human learning process and external behavior in the context of game theory. Kimble (1961) defined learning as a change in human behavior over a relatively long period through observations, cognitions, and exercises. Specifically, psychologists consistently agree that learning itself has not been directly observed (see Brenner, 1999). However, learning is still studied extensively in psychology and in the social sciences.

2.2 Portfolio Selections

A main issue in finance is analyzing individuals' asset allocations for the purpose of maximizing their terminal utility of consumption or wealth. Samuelson (1969) and Merton (1969, 1971) first developed a continuous-time decision model to solve individual optimal portfolio selection problems. They provided an explicit solution for optimal stock investments using a stochastic optimal control methodology. Framed in a perfect economic system, many studies have obtained a simple fixed-mixed strategy for the asset allocations of representative investors with a constant relative risk aversion utility (see Campbell and Viceira, 2002; Hens and Wöhrmann, 2007; Antoine, 2012). Numerous sequent researchers have extended the Samuelson-Merton models to analyze related financial issues.

2.3 Portfolio Choices and Learning

Numerous studies have concerned themselves with the learning features of human investment behavior. For example, it was found that individuals' purchasing decisions were often related to others' decisions through the observations of signals or behaviors (Bihkchandani et al., 1998). In other words, investors could learn something from observing the results of other's decisions. Brennan (1998) analyzed how investors learned appropriate rules about the mean return of a risky asset and could increase or decrease their holding weights of the risky asset through their estimations, given a long horizon. For reducing the uncertainty of the risky asset price, investors could form a demand for hedging the estimated risk of the return parameters.

Xia (2001) specified a stochastic volatility model for numerous predictive variables regarding mean returns and state variables. Sharpe (2010) proposed an adaptive asset allocation policy by considering market movements for outstanding mutual funds. Such a policy could reduce the contrarian behavior of a majority of investors.

Other studies have involved a learning feature in asset allocation issues. Brandt et al. (2005) developed a simulation-based method for solving discrete-time decision problems of portfolio choices. The method could handle some assets with arbitrary return distributions and state variables with path-dependent dynamics. Thus, the researchers took into account the return predictabilities but were uncertain about the parameters of the data generating process in their method for analyzing how investors predict, learn, and estimate these parameters of stock return and state variables. Cvitanić et al. (2006) also solved an optimal portfolio decision problem for a non-myopic utilitarian investor who has incomplete information regarding an abnormal expected return from risky stock. Their results indicated that a hedging demand for risky assets included a learning component toward the abnormal expected returns. Hens and Wöhrmann (2007) proposed a recurrent reinforcement learning method for investing in several international stock markets framed in a non-constant investment opportunity set. Their empirical study found that the optimal asset allocations noticeably differed from the fixed-mixed rule. Branch and Evans (2010) and Guidolin and Timmermann (2007) focused on expectation formations that incorporated several issues in behavioral finance and considered regime-switching returns and volatilities in their learning formulations.

In this study, we attempt to solve the individual's decisions regarding portfolio selections as between risky stocks and risk-free bonds when the individual has an inherent learning ability. We are concerned with how the individual's learning process regarding his market cognition changes these portfolio choices. Specifically, we answer the question of whether or not portfolio choices involving a learning process in terms of market cognition regarding *volatility risk* differ from choices involving a learning process in terms of market cognition regarding *jump risk*.

3. Portfolio Choices and Learning Effect

This study develops a continuous-time decision model for an individual's portfolio selections, considers his learning ability, and examines how learning changes his optimal asset allocations. The study assumes that a financial market is composed of two representative assets. For simplicity of analysis and without the loss of generality, we further assume that our analytical model has the following conditions. First, the financial market is perfect, transparent, efficient, and frictionless, and there are no transaction costs or taxes. Second, short trading is allowed, and the assets are infinitesimally divisible. Third, a representative investor dynamically manages his asset allocations to maximize his lifetime utility of wealth capital under a continuous-time stochastic framework.

The first asset is a risk-free bond that pays investors a constant rate of interest. The price dynamic ($M(t)$) of the risk-free bond is governed as follows:

$$dM(t) = rM(t)dt, \quad M(t=0) = 1, \quad (1)$$

where $r \geq 0$ denotes the interest rate of the risk-free bond.

The second asset is a risky asset represented by a common stock that is non-dividend paying. The price dynamic ($S(t)$) of the stock follows a mixed Poisson-geometric Brownian motion and is described as follows:

$$dS(t) = [r + kV(t) - \theta\lambda V(t)]S(t)dt + \sqrt{V(t)}S(t)dZ_s(t) + Y(t)S(t-)dQ(t), \quad (2)$$

$$S(t=0) = S_0,$$

$$dV(t) = [m - nV(t)]dt + \sigma\sqrt{V(t)}dZ_v(t), \quad V(t=0) = V_0, \quad (3)$$

where dZ_s and dZ_v are the increments of a Wiener process with correlation ρ_{sv} , V denotes the stochastic instantaneous variance of stock returns, Q denotes a Poisson process with intensity λV , and $Y(t)$ denotes a random jump in stock price with a mean θ . To impose a non-negative condition, we assume that $Y \in [-1, \infty)$. Parameters k , m , n , σ , and λ are positive constants. Equation (2) states the price dynamic of the stock whose risks originate from instantaneous minor volatilities and instantaneous large jumps, captured by geometric Brownian motion and a Poisson process, respectively. Specifically, the volatility rate of the risky asset is stochastic and is described by (3). Reflecting the two types of risk sources of the stock price, the mean return term presents two risk premiums, kV and $\theta\lambda V$. Many studies in theoretical finance have the same settings as dynamic system (2) and (3) (e.g., Merton, 1980; Pan, 2002; Liu et al., 2003).

The decision-maker who has an innate learning ability can observe the current situations of financial markets and can form his own cognitions that underpin his investment rules. According to learning psychology, learning is a gradual cumulative process in response to varied external information. We describe the learning process in response to market information as being a mixed Poisson process and geometric Brownian motion. Specifically, the learning process can advance quickly as a result of a high level of cognition regarding stochastic jumps in stock risk. Generally speaking, individuals can immediately learn more as they are exposed to a sudden jump risk; thus, an individuals' learning process can immediately be enhanced through an observation of the jump risks of stock returns. Consequently, this study regards the learning process as being a state variable affecting asset allocations in which the dynamic process of the individual's learning about market risks follows a mixed Poisson process and geometric Brownian motion. Thus, learning has the feature of steady growth in the long run, but it can also grow quickly as some sudden events occur. For example, as the individual is exposed to market collapses or sudden important events resulting in a movement, he will immediately learn, greatly enhancing his cognition regarding price behaviors or market phenomena in the process. In short, we can describe this discrete and sudden learning cognition with a Poisson jump process. Otherwise, the individual's learning process is varied and steady given an instantaneous period; consequently, we describe this part of the

learning process with geometric Brownian motion. Incorporating these two processes into a single learning process is appropriate for reflecting individual market cognition.

In this study, the learning accumulation process is based on individual cognition of the stochastic volatility risk and jump risk of stock returns. That is to say, a change in the stochastic risks brings about a change in individual cognition, especially at the time of sudden jumps in prices.

Incorporating the above two types of learning cognition regarding the volatility risk and jump risk of stock returns, we set the dynamic process of the individual's learning as follows:

$$\begin{aligned} dL &= [\alpha + \eta\lambda V(t)]L(t)dt + \sqrt{V(t)}\xi(t)dZ_L(t) + X(t-)dQ(t), \\ L(t=0) &= L_0, \end{aligned} \tag{4}$$

where α denotes a long-term growth rate of the learning process, $X(t) \in [0, \infty)$ denotes a random jump size in individual learning with an intensity λ and reflects individual cognition regarding the jump risks of stock markets, η denotes a premium for increasing one's learning ability as a sudden jump event occurs, and ξ denotes instantaneous volatility of the learning process and represents cognition regarding the volatility risk of stock markets. That is, the individual's market risk cognition is reflected in his dynamic learning process. The jumps in the stock price ($Y(t)$) and the jumps in the individual learning ($X(t)$) are independent of each other and of the three Wiener processes $Z_s(t)$, $Z_v(t)$, and $Z_L(t)$. However, the three Wiener processes themselves have constant correlations with $E(dZ_s dZ_L) = \rho_{sL} dt$, $E(dZ_v dZ_s) = \rho_{sv} dt$, and $E(dZ_v dZ_L) = \rho_{Lv} dt$.

The individual implements his decision-making about portfolio choices in the above-described economic system and learning process setting. Incorporating the individual's asset investments, we obtain his wealth accumulation as follows:

$$\begin{aligned} dW(t) &= [r + w(t)(k - \theta\lambda)V(t) + \pi L(t)]W(t)dt + \sqrt{V} w(t)W(t)dZ_s \\ &\quad + Y(t)w(t)W(t)dQ, \\ W(t=0) &= W_0, \end{aligned} \tag{5}$$

where $\pi > 0$ is a learning premium for the individual who has a cognitive ability regarding market conditions, since the individual's learning ability can benefit his wealth accumulation. Here $w(t)$ denotes the fraction represented by the risky asset with respect to the overall wealth capital. All other wealth is invested in risk-free bonds. Therefore, the income sources for the accumulation process of wealth capital are the risky assets, cash accounts, and learning premiums. The risk sources for the accumulation process of wealth capital originate from the instantaneous minor volatilities and large sudden jumps in stock price.

The objective of the representative individual is to maximize the lifetime utility of his wealth capital as follows:

$$E_0 \left[\int_{t=0}^{t=T} e^{-\rho t} U(W(t), t) dt + e^{-\rho T} B(W(T), T) \right], \quad (6)$$

where ρ denotes a discount rate of direct utility functions $U(\cdot)$ and bequest functions $B(\cdot)$. The direct utility function is assumed to be strictly increasing, concave, and twice continuously differentiable with respect to wealth and time t . Specifically, it satisfies the Inada (1963) conditions:

$$\lim_{W \rightarrow 0} U_w(W(t), t) = \lim_{t \rightarrow 0} U_t(W(t), t) = \infty, \text{ and}$$

$$\lim_{W \rightarrow \infty} U_w(W(t), t) = \lim_{t \rightarrow T} U_t(W(t), t) = 0.$$

Here $E_t[\cdot]$ denotes an expectation operator conditional on information sets at time t .

Subject to the dynamic wealth process (5), the individual implements a portfolio choice to maximize his expected utility during the period $0 \leq t \leq T$. The state equations of the decision problem are the wealth capital dynamic (5), the stochastic variance dynamic (3), and the learning process (4). The control variables are the weights of the risky asset ($w(t)$).

According to a stochastic dynamic programming methodology, an indirect utility function for the problem is as follows:

$$J(W(t), V(t), L(t), t) = \max_{\{w(t), 0 \leq t \leq T\}} E_0 \left[\int_0^T e^{-\rho t} U(W(t), t) dt + e^{-\rho T} B(W(T), T) \right], \quad (7)$$

where $J(W(t), V(t), L(t), t)$ is twice continuously differentiable, strictly increasing, and concave with respect to time, wealth capital, and stochastic variance.

Using Bellman's principle of optimality, we can obtain the following Hamilton-Jacobi-Bellman (henceforth, HJB) equation:

$$0 = \max_{w, C} [J_t + DJ + U(W(t), t)], \quad (8)$$

where $D(\cdot)$ is a Dynkin differential operator, which implies that:

$$\begin{aligned} DJ &= J_w [w(k - \theta\lambda)V + \pi L - C]W + \frac{1}{2} J_{ww} Vw^2W^2 + J_v (m - nV) \\ &+ \frac{1}{2} J_{vv} \sigma^2 V + J_L (\alpha + \eta\lambda V)L + \frac{1}{2} J_{LL} VL^2 + J_{vw} V\sigma wW\rho_{SV} \\ &+ J_{wL} VLwW\rho_{SL} + J_{LV} \sigma VL\rho_{LV} \\ &+ \lambda V [E(J(W(1+wY), V, L+X, t)) - J] \\ &= 0. \end{aligned}$$

where J_w , J_v , J_{vv} , J_L , J_{LL} , J_{vw} , J_{LV} , J_{LW} , J_t , and J_{ww} denote partial derivatives of $J(W, V, L, t)$ with respect to W , V , L , and t . To obtain an explicit solution of the optimal strategies of portfolio choices, we set both direct and indirect utility functions as follows:

$$U(W(t),t) = \frac{W(t)^{1-\gamma}}{1-\gamma}, \tag{9}$$

$$J(W, V, L, t) = \frac{1}{1-\gamma} W^{1-\gamma} \exp[a(t) + b(t)V(t) + c(t)L(t)], \tag{10}$$

where $\gamma > 0$ denotes a relative risk aversion measure proposed by Arrow-Pratt, and $a(t)$, $b(t)$, and $c(t)$ are the risk parameters that change over time. The utility functions have a form of constant relative risk aversion (CRRA).

Our problem is to find the optimal weights of the stock asset in the investor's portfolio. Taking the first-order partial derivatives for the stock weight (w), we derive the following corollary.

Corollary: Subject to the stochastic variance process (3) and the wealth capital constraint (5), an individual who has an innate learning process (4) in response to market risks and whose utility function follows CRRA forms (9) and (10) has the following optimal stock holding to maximize his expected utility:

$$w^* = \frac{k - \theta\lambda}{\gamma} + \frac{c\xi\rho_{SL}}{\gamma} + \frac{b\rho_{SV}\sigma}{\gamma} + \frac{\lambda Y(1 + w^*Y)^{-\gamma} \exp(cX(t))}{\gamma}, \tag{11}$$

where $a(t)$, $b(t)$, and $c(t)$ can be obtained by following differential equations.

$$a'(t) + mb(t) + c(t)\alpha + (1-\gamma)r = 0 \tag{12}$$

$$b' + \frac{b^2\sigma^2}{2} + [\sigma w^* \rho_{SV}(1-\gamma) - n]b + [\xi w^* \rho_{SL}(1-\gamma) - \eta\lambda]c + \frac{c^2\xi^2}{2} + bc\sigma\xi\rho_{LV} + \left(\frac{-\gamma(1-\gamma)w^{*2}}{2} + (k - \theta\lambda)(1-\gamma)w^* + \lambda((1 + w^*)^{1-\gamma} \exp(cX) - 1) \right) = 0 \tag{13}$$

$$\frac{c'(t)}{1-\gamma} + \pi = 0. \tag{14}$$

Proof: See Appendix.

The corollary provides numerous interesting results, in which the optimal holding fraction of the stock relates to the mean growth rate, the volatility rate of stock returns, and numerous others factors, as shown in (11). The optimal weight (w) does not depend on the state variables W , V , and L . That is, the optimal stock holding is not affected by a market timing effect (see Liu et al., 2003). Consequently, we have solved a closed-form solution of the optimal weight of the stock. The risk parameters $a(t)$, $b(t)$, and $c(t)$ can be derived through a numerical analysis of (12)–(14).

The determinants of the individual's stock holding (w) can be divided into four demands. The first term is a self-investment demand that reflects the impacts of the expected excess return (k), the risk premium of the jump in prices (θ), and the

default probability (λ). The individual will invest more wealth capital in the stock market if the net expected return rate (i.e., $k - \theta\lambda$) is higher. The second term is a demand for the stock in response to the individual's innate learning ability. As the relations between the stock and the learning process appear to be positive, the individual will tend to favor more stock as his learning accumulates. If, through observing stock markets, the individual forms his cognitions as part of his learning process, he will learn more about the movements of the stock market and will allocate his wealth capital in a way consistent with his learning process. The third term is the hedging demand against the stochastic volatility risk. The individual would like to increase or decrease his stock holdings to hedge against the risk associated with the stochastic volatility based on the correlations ρ_{sv} . The fourth term is the demand for illiquidity hedging (see Liu et al., 2003). This demand reflects insufficient liquidity as the stock price carries a sudden jump risk. Specifically, the demand for illiquidity hedging involves the individual's learning process in terms of how individual cognition reacts to the jump risk (i.e., X).

We specifically focus on the second term, the stock demand, which relates to the individual's learning. Given a positive relationship between the stock price process and the learning process, the result implies that as the individual improves his learning ability in response to the stock price movements, he will hold more stock due to his learning accumulation process. Therefore, our results differ from the optimal portfolio choice decision solutions found in traditional studies which do not consider an individual's learning process and consider only the estimations or predictions of uncertain parameters. Our study reflects how the investors' innate learning process affects his optimal decision-making in terms of portfolio selections.

Furthermore, ignoring the jump risk of the stock price, we can obtain the following equation:

$$k + c\xi\rho_{sl} - \gamma w^* + b\sigma\rho_{sv} = 0. \quad (15)$$

This equation expresses a first-order condition of optimal portfolio choices. That is the optimal condition of the marginal benefit equaling the marginal cost for holding a risky asset. Compared to the Merton (1971, 1990) portfolio selection model, in addition to the premium regarding the stochastic volatility, we also find that the stock provides a learning premium regarding the individual's cognition about the stock markets. In short, both the stochastic volatility risk and the learning effect should be priced in a pricing model of equity assets because both stochastic volatilities and an individual's learning process appear to be important in determining stock prices.

4. Numerical Examples

To identify how the learning process affects investors' optimal portfolio choices, this study implements several numerical analyses. We first examine the learning process in terms of how the individual's cognitions (i.e., X and ξ) in response to market risks change his asset allocations. Table 1 shows several panels detailing the

optimal holdings of the stock, including learning factors, market factors, and risk premium factors. Panel A displays how the optimal weight of stocks is changed by two learning parameters regarding the individual's cognitions of market volatility risk and jump risk. Here, the purpose of the examination is to observe how the individual's learning ability, in terms of tracking market risks, changes his asset allocations. Thus, we can answer the question of whether or not an individual with a strong learning ability regarding stock market conditions will increase his stock holdings.

As shown in Panel A, the optimal holding is negatively related to the cognitive parameter (ξ) in the individual's learning process. That is, the individual with an increasingly sensitive cognitive ability regarding the volatility risk of the stock market will invest less in stocks. One possible explanation is that the risk-averse individual can appropriately and increasingly reflect the volatility risk through a decrease in investment in the risky asset as his risk-cognition regarding volatility risk increases, which hedges against the volatility risk. Also, the optimal stock holdings negatively relate to the cognitive parameter (X) in his learning process. If the individual can perceptively detect the jump risk in stock markets, he will hold less stocks. That is to say, the individual tends to avoid the jump risks based on his strong cognition regarding the stock markets. Compared to stock holdings for an individual without learning ability, the individual with a risk-cognition ability reduces his risky asset investments since he wants to avoid the adverse impacts of both volatility risks and jump risks of which he is cognizant. An individual without risk-cognition ability will put too much of his wealth in stocks and consequently be over-exposed to risk.

Next, we analyze the impacts of the market factors. Panel B presents the relations between the optimal portfolio choices and the jump probability (λ) of stock market risks and the instantaneous volatility rate (σ) of the stochastic variance process. An increase in the jump probability (λ) can cause an increase in the individual's stock holdings. Here, the results conflict with a common result found in previous studies that did not consider a learning effect. That is to say, the individual in this study chooses to allocate more wealth on stocks rather than on risk-free bonds because he has a cognitive ability regarding stock jump risks and his cognition of jump risk can enhance his asset allocations via a learning process. In addition, if the volatility rate of the stock variance process increases, the individual will reduce his stock holdings to hedge against the volatility risk of the stochastic variance process. In Panel B, we see that, although the sensitivities of stock holdings are lower, both the jump probability and the volatility risk have changed the holding rates of the stock.

Turning to the risk premium factor (θ) in Panel C, we find that the holding ratios decrease if the adverse jump impact increases in terms of a decrease of the mean growth rate of the stock return. Therefore, the stock mean return decreases (i.e., risk premium factor θ increases) if the adverse jump event occurs, and the individual tends to reduce his stock holding to hedge against the jump risk.

We also examine how the degree of correlation between the stock price process and the learning process changes the individual's optimal portfolio choices. A higher ρ_{SL} denotes that the individual has a better ability to be cognizant of market conditions, while a lower ρ_{SL} implies that the individual is less aware of market conditions in

terms of his learning process. We find that the individual holds less stock if the above correlation increases. That is to say, an individual will hold less stock if he has a strong learning ability in terms of tracking the stock market. Previous studies that do not consider an innate learning ability appear to show a bias in their results in terms of optimal strategy for portfolio choices since they suggest holding ratios of risky assets that are too high. Thus, we have answered whether an individual with a strong learning ability regarding stock market conditions will increase his stock holdings.

Table 1. Optimal Portfolio Choice and Learning Process in Response to Market Conditions

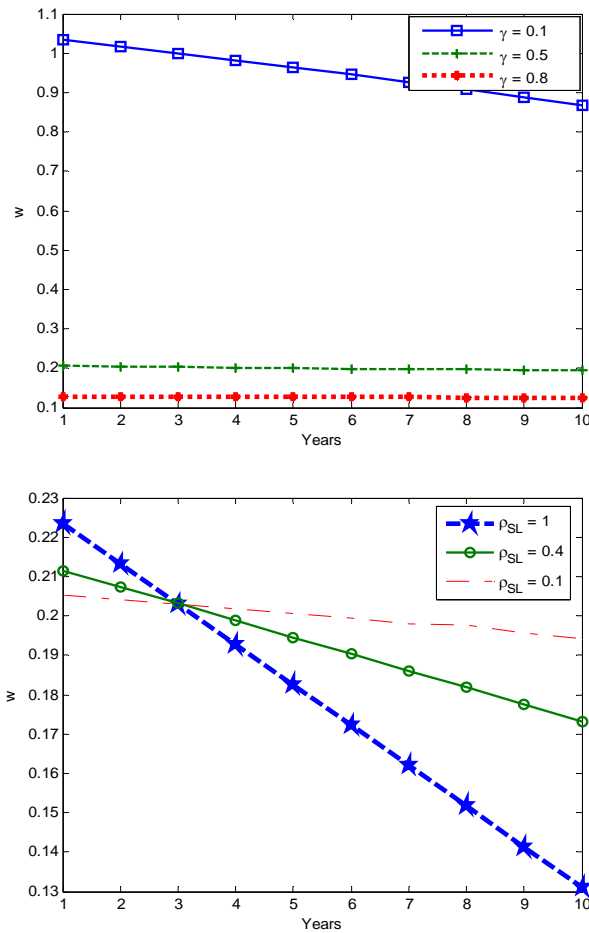
Panel A: Learning factors		X				
ξ	0.1	0.2	0.3	0.4	0.5	0.6
0.05	0.2014	0.2013	0.2013	0.2012	0.2012	0.2012
0.10	0.2007	0.2006	0.2006	0.2005	0.2004	0.2004
0.15	0.1999	0.1998	0.1998	0.1998	0.1997	0.1997
0.20	0.1994	0.1994	0.1993	0.1992	0.1991	0.1990
0.25	0.1984	0.1983	0.1983	0.1983	0.1982	0.1982
0.30	0.1976	0.1976	0.1975	0.1975	0.1974	0.1974
Panel B: Market factors		λ				
σ	0.001	0.006	0.011	0.016	0.021	0.026
0.05	0.1971	0.1985	0.1999	0.2013	0.2027	0.2041
0.10	0.1968	0.1983	0.1996	0.2010	0.2024	0.2037
0.15	0.1966	0.1980	0.1994	0.2007	0.2021	0.2034
0.20	0.1964	0.1977	0.1991	0.2004	0.2018	0.2031
0.25	0.1962	0.1975	0.1988	0.2002	0.2015	0.2028
0.30	0.1960	0.1973	0.1986	0.1999	0.2012	0.2025
Panel C: Risk premium factors		θ				
ρ_{sl}	0.01	0.02	0.03	0.04	0.05	0.06
1.0	0.1731	0.1729	0.1727	0.1725	0.1723	0.1720
0.8	0.1791	0.1789	0.1787	0.1785	0.1783	0.1781
0.6	0.1852	0.1849	0.1847	0.1845	0.1843	0.1841
0.4	0.1912	0.1910	0.1908	0.1905	0.1903	0.1901
0.2	0.1972	0.1970	0.1968	0.1665	0.1963	0.1961
0.0	0.2032	0.2030	0.2028	0.2026	0.2024	0.2023

Notes: This table presents the relationships between several factors and the holding ratios of the stock. The parameters are set as follows: $\kappa = 0.1$, $\pi = 0.05$, $\alpha = 0.1$, $\sigma = 0.15$, $\rho_{sl} = 0.1$, $\rho_{LV} = 0.1$, $\rho_{SV} = 0.1$, $t = 6$, $\gamma = 0.05$, $\xi = 0.2$, $X = 0.2$, $Y = 0.2$, $m = 0.1$, $n = 0.05$, $\eta = 0.05$, $\lambda = 0.011$, $\theta = 0.05$.

Next we further analyze how the individual with a learning ability adjusts his portfolio selections over time as shown in Figure 1. We first observe the optimal strategies of individuals with various risk aversion attitudes. This study sets an investment horizon that varies from 1 to 10 years. We find that an individual with a lower risk aversion attitude (i.e., $\gamma = 0.1$) holds more stock; however, the holding rate gradually decreases over time. Otherwise, an individual with a higher risk aversion

attitude tends to hold less stock, and the change in terms of the holding rate is more stable. That is, an individual with a lower risk aversion adjusts his portfolio choices quickly over time compared to an individual with a higher risk aversion. Generally, individuals gradually allocate less of their wealth towards risky assets when the investment horizon increases. These results imply that an individual tends to avoid the risk of holding a long-horizon investment.

Figure 1. Optimal Portfolio Choice and Investment Horizon



Notes: This figure plots the optimal portfolio choices varying with investment time. The parameters are set as follows: $\kappa = 0.1$, $\pi = 0.05$, $\alpha = 0.1$, $\sigma = 0.15$, $\rho_{SL} = 0.1$, $\rho_{LV} = 0.1$, $\rho_{SV} = 0.1$, $\gamma = 0.05$, $\xi = 0.2$, $X = 0.2$, $Y = 0.2$, $m = 0.1$, $n = 0.05$, $\eta = 0.05$, $\lambda = 0.011$, $\theta = 0.05$.

We also observe the effects of the correlations between the stock and the learning process in terms of a longer horizon. We find that an individual holds more stock if the correlation is higher (i.e., he has a strong learning ability) in a short horizon. The

results indicate that an individual whose learning process correlates strongly with stock movements would like to invest more in stocks. However, these stock holdings quickly decrease over time. This implies that the individual tends to become more conservative over time.

5. Conclusion

This study provides an optimal decision-making analysis in terms of portfolio selections for an individual with an innate learning ability in order to answer the question of *how* this innate learning process changes his asset allocations in a continuous-time setting. Thus, we provide a closed-form solution for the problem of optimal portfolio selections for an individual who has an innate learning ability which allows him to cognize the market risks of stock investments. Compared to the traditional results produced by the Samuelson-Merton model, this study provides innovative observations purely regarding the learning factor impact on optimal asset allocation decisions.

Our specific findings are as follows. First, individual stock holdings are positively related to the correlation between stock price and the learning process. Second, regardless of the individual's risk aversion attitude or the correlations between stock price and the learning process, although a low risk-averse individual with a strong learning ability individual will choose to hold more stocks, his stock holdings decrease gradually over time. Third, considering the individual's learning process in terms of risk-cognition regarding both volatility risks and jump risks, the individual allocates less wealth to stock holdings than to risk-free bonds. Finally, the study resolves the Samuelson-Merton portfolio choice problem framed in a fixed investment opportunity set for an individual with a learning ability about risk cognition and suggests holding a lower fraction of risky stocks compared to the results published by previous authors.

Appendix

This appendix gives a proof for the optimal holdings of the stock for an individual with an innate learning ability. First, we differentiate the HJB equation (8) with respect to the stock weight and obtain the following:

$$\begin{aligned} \psi_w = J_w W [(k - \theta\lambda)V] + J_{ww} W^2 wV + J_{vw} V \sigma W \rho_{sv} + J_{wL} VW \xi \rho_{sL} \\ \lambda V Y W J_w (W(1 + wY), V, L + X, t) = 0. \end{aligned} \quad (A1)$$

In addition, we differentiate the indirect utility function (10) with respect to the wealth capital, stochastic variance, and learning accumulation to obtain yield the following:

$$\begin{aligned} J_w = W^{-\gamma} \exp[a(t) + b(t)V(t) + c(t)L(t)], \\ J_{ww} = -\gamma W^{-\gamma-1} \exp[a(t) + b(t)V(t) + c(t)L(t)], \end{aligned}$$

$$\begin{aligned}
 J_V &= \frac{1}{1-\gamma} b(t)W^{1-\gamma} \exp[a(t) + b(t)V(t) + c(t)L(t)], \\
 J_{VV} &= \frac{1}{1-\gamma} b^2(t)W^{1-\gamma} \exp[a(t) + b(t)V(t) + c(t)L(t)], \\
 J_L &= \frac{1}{1-\gamma} c(t)W^{1-\gamma} \exp[a(t) + b(t)V(t) + c(t)L(t)], \\
 J_{LL} &= \frac{1}{1-\gamma} c^2(t)W^{1-\gamma} \exp[a(t) + b(t)V(t) + c(t)L(t)], \\
 J_{VL} &= \frac{1}{1-\gamma} b(t)c(t)W^{1-\gamma} \exp[a(t) + b(t)V(t) + c(t)L(t)], \\
 J_{SV} &= bW^{-\gamma} \exp[a(t) + b(t)V(t) + c(t)L(t)], \\
 J_{SL} &= cW^{-\gamma} \exp[a(t) + b(t)V(t) + c(t)L(t)].
 \end{aligned}$$

Therefore, the risk aversion measures can be rewritten as follows:

$$\begin{aligned}
 \frac{-J_W}{J_{WW}W} &= \frac{1}{\gamma}, \\
 \frac{-J_{VW}}{J_{WW}W} &= \frac{b(t)}{\gamma}.
 \end{aligned}$$

Taking the above risk aversion measures into (A1) then yields optimal strategies (11) of the risky asset. Furthermore, (11) can be substituted into HJB equation (10), after which the following is obtained:

$$\begin{aligned}
 &\frac{1}{1-\gamma} W^{1-\gamma} e^{(a(t)+b(t)V+c(t)L)} + W^{1-\gamma} e^{(a(t)+b(t)V+c(t)L)} (r + w(\kappa - \theta\lambda)V + \pi L) \\
 &- \frac{1}{2} \gamma W^{1-\gamma} e^{(a(t)+b(t)V+c(t)L)} V W^2 + \frac{1}{1-\gamma} W^{1-\gamma} e^{(a(t)+b(t)V+c(t)L)} b(m - nV) \\
 &+ \frac{1}{2} \frac{1}{1-\gamma} W^{1-\gamma} b^2 e^{(a(t)+b(t)V+c(t)L)} \sigma^2 V + \frac{1}{1-\gamma} W^{1-\gamma} c e^{(a(t)+b(t)V+c(t)L)} (\alpha - \eta\lambda V) \\
 &+ \frac{1}{2} \frac{1}{1-\gamma} W^{1-\gamma} c^2 e^{(a(t)+b(t)V+c(t)L)} \xi^2 V + W^{1-\gamma} b e^{(a(t)+b(t)V+c(t)L)} \sigma V w \rho_{SV} \\
 &+ W^{1-\gamma} c e^{(a(t)+b(t)V+c(t)L)} \xi V w \rho_{SL} + \frac{1}{1-\gamma} W^{1-\gamma} b c e^{(a(t)+b(t)V+c(t)L)} \sigma V \xi \rho_{LV} \\
 &+ \lambda V e^{(a(t)+b(t)V+c(t)L)} \left(\frac{1}{1-\gamma} \right) W^{1-\gamma} [(1 + wY)^{1-\gamma} e^{cX} - 1] = 0
 \end{aligned} \tag{A2}$$

Setting the constant term of (A2) to zero, we obtain (12). Equation (13) is obtained if we set the coefficient term of stochastic variance (V) in (A2) to zero. In addition, (14)

can be obtained through the coefficient term of the learning process (L).

Notes

1. Although this study assumes that all parameters remain constant, the portfolio selections are still analyzed in a stochastic environment in which individuals are exposed to both stochastic volatility risks and jump risks of stock returns.

References

- Antoine, B., (2012), "Portfolio Selection with Estimation Risk: A Test-Based Approach," *Journal of Financial Econometrics*, 10, 164-197.
- Basak, S. and G. Chabakauri, (2010), "Dynamic Mean-Variance Asset Allocation," *Review of Financial Studies*, 23, 2970-3016.
- Bawa, V. S., S. J. Brown, and R. W. Klein, (1979), *Estimation Risk and Optimal Portfolio Choice*, New York: North-Holland.
- Bikhchandani, S., D. Hirshleifer, and I. Welch, (1998), "Learning from the Behavior of Others: Conformity, Fads, and Informational Cascades," *Journal of Economic Perspectives*, 12, 151-170.
- Börgers, T., (1996), "On the Relevance of Learning and Evolution to Economic Theory," *The Economic Journal*, 106, 1374-1385.
- Börgers, T. and R. Sarin, (1997), "Learning through Reinforcement and Replicator Dynamics," *Journal of Economic Theory*, 77, 1-14.
- Brandt, M. W., A. Goyal, P. Santa-Clara, and J. R. Stroud, (2005), "A Simulation Approach to Dynamic Portfolio Choice with an Application to Learning about Return Predictability," *Review of Financial Studies*, 18, 831-873.
- Branch, W. A. and G. W. Evans, (2010), "Asset Return Dynamics and Learning," *Review of Financial Studies*, 23, 1651-1680.
- Brennan, M. J., (1998), "The Role of Learning in Dynamic Portfolio Decisions," *European Finance Review*, 1, 295-306.
- Brennan, M. J. and Y. H. Xia, (2002), "Dynamic Asset Allocation under Inflation," *Journal of Finance*, 57, 1201-1238.
- Brenner, T., (1999), *Modeling Learning in Economics*, Edward Elgar.
- Campbell, J. Y. and L. M. Viceira, (2002), *Strategic Asset Allocation*, Oxford University Press.
- Chellathurai, T. and T. Draviam, (2007), "Dynamic Portfolio Selection with Fixed and/or Proportional Transaction Costs Using Non-Singular Stochastic Optimal Control Theory," *Journal of Economic Dynamics and Control*, 31, 2168-2195.
- Cvitanić, J., A. Lazrak, L. Martellini, and F. Zapatero, (2006), "Dynamic Portfolio Choice with Parameter Uncertainty and the Economic Value of Analysts' Recommendations," *Review of Financial Studies*, 19, 1113-1156.
- Detemple, J. B., (1986), "Asset Pricing in a Production Economy with Incomplete Information," *Journal of Finance*, 41, 383-391.
- Dothan, M. U. and D. Feldman, (1986), "Equilibrium Interest Rates and Multiperiod

- Bonds in a Partially Observable Economy,” *Journal of Finance*, 41, 369-382.
- Guidolin, M. and A. Timmermann, (2007), “Asset Allocation under Multivariate Regime Switching,” *Journal of Economic Dynamics & Control*, 31, 3503-3544.
- Hens, T. and P. Wöhrmann, (2007), “Strategic Asset Allocation and Market Timing: A Reinforcement Learning Approach,” *Computational Economics*, 29, 369-381.
- Inada, K. I., (1963), “On a Two-Sector Model of Economic Growth: Comments and a Generalization,” *Review of Economic Studies*, 30, 119-127.
- Kimble, G. A., (1961), *Hilgard and Marquis’ Conditioning and Learning*, Englewood Cliffs: Prentice Hall.
- Liu, J., F. Longstaff, and J. Pan, (2003), “Dynamic Asset Allocation with Event Risk,” *Journal of Finance*, 58, 231-259.
- Merton, R. C., (1969), “Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case,” *The Review of Economics and Statistics*, 51, 247-257.
- Merton, R. C., (1971), “Optimum Consumption and Portfolio Rules in a Continuous-Time Model,” *The Journal of Economic Theory*, 3, 373-413.
- Merton, R. C., (1980), “On Estimating the Expected Return on the Market: An Exploratory Investigation,” *Journal of Financial Economics*, 8, 323-361.
- Merton, R. C., (1990), *Continuous-Time Finance*, Oxford: Blackwell Press.
- Mitropoulos, A., (2004), *Economic Learning, Experiments and the Limits to Information*, Edward Elgar.
- Pan, J., (2002), “The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study,” *Journal of Financial Economics*, 63, 3-50.
- Samuelson, P. A., (1969), “Lifetime Portfolio Selection by Dynamic Stochastic Programming,” *The Review of Economics and Statistics*, 51, 239-246.
- Sharpe, W. F., (2010), “Adaptive Asset Allocation Policies,” *Financial Analysts Journal*, 66, 45-59.
- Shiraishi, H., (2012), “A Simulation Approach to Statistical Estimation of Multiperiod Optimal Portfolios,” *Advances in Decision Sciences*, forthcoming.
- Simon, H. A., (1956), *Administrative Behavior*, New York: The Macmillan Company.
- Xia, Y., (2001), “Learning about Predictability: The Effects of Parameter Uncertainty on Dynamic Asset Allocation,” *Journal of Finance*, 56, 205-246.