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A Note on Equivalences in Measuring Returns to Scale

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In this note we derive necessary and sufficient condition for equivalence of input oriented and output oriented scale elasticity measures for multi-output, multi-input technologies and provide a Lagrange multiplier (or shadow price) interpretation of these scale elasticity measures.

Following Färe et al. (1986), recall that one can measure local returns to scale via the *output oriented measure of scale elasticity*, defined as:

$$e_o(x,y) = \left.\frac{\partial \ln \theta}{\partial \ln \lambda}\right|_{\theta=1,\lambda=1}, \text{ such that } D_o(\lambda x, \theta y) = 1. \tag{1}$$

or, alternatively, via the *input oriented measure of scale elasticity*, defined as:

$$e_i(y,x) \equiv \left.\frac{\partial \ln \lambda}{\partial \ln \theta}\right|_{\theta=1,\lambda=1}, \text{ such that } D_i(\theta y, \lambda x) = 1. \tag{2}$$

where x and y are vectors of inputs and outputs, respectively, while $D_i(y,x)$ and $D_o(x,y)$ are Shephard's (1957, 1970) input and output distance functions, respectively. To save space here, we refer to Färe and Primont (1995) for the notation, definitions and properties of the functions involved.[1, 2]

Meanwhile, note that both formulas are trying to measure scale elasticity by looking at the relationship between *equi-proportional* changes in *all* inputs with *equi-proportional* changes in *all* outputs, but they do it by using different characterizations of technology. Thus, several natural questions arise: "What is a relationship between these two alternative measures of scale elasticity?" Are (1) and (2) equivalent? Always? Under what conditions? Färe et al. (1986) were the first to provide a version of this result with a proof. In the theorem below, we revisit this result and provide a Lagrange multiplier interpretation of it.

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Theorem. Given definitions above, standard regularity conditions of production theory and assuming that, in a neighborhood of a point of interest, the distance functions $D_i(y,x)$, $D_i(y,x)$, and $D_o(x,y)$, $D_o(x,y)$ are continuously differentiable with respect to each element of (x,y), we have:

$$e_o(x,y) = 1/e_i(y,x)\,, \tag{3}$$

if and only if

$$\nabla'_y D_i(y,x)y \neq 0 \text{ and } \nabla'_x D_o(x,y)x \neq 0\,. \tag{4}$$

Proof. Due to the implicit function theorem, we can rewrite (1) and (2) in a more compact form:

$$e_o(x,y) = -\nabla'_x D_o(x,y)x \text{ and } e_i(y,x) = -\nabla'_y D_i(y,x)y\,. \tag{5}$$

To prove necessity, assume that $e_o(x,y) = 1/e_i(y,x)$, then (5) implies that:

$$\nabla'_x D_o(x,y)x = 1/\nabla'_y D_i(y,x)y\,,$$

which can hold only if (4) is true. To prove sufficiency, assume (4) is true. Further, note that given standard regularity conditions, we can rewrite the output distance function as follows:

$$D_o(x,y) = \inf\{\theta > 0 : D_i(y/\theta, x) \geq 1\}\,. \tag{6}$$

The resulting Lagrangian function for this optimization problem can be written as:

$$L(\theta,\mu \mid x,y) = \theta - \mu(D_i(y/\theta,x) - 1)\,, \tag{7}$$

and so the system of equations defined by the first order condition for (7) is given by:

$$\nabla_\theta L \Big|_{\substack{\theta=\theta^* \\ \mu=\mu^*}} = 1 - \mu^* \nabla'_{y/\theta} D_i(y/\theta^*, x)y(-1/(\theta^*)^2) = 0\,, \tag{8}$$

and

$$\nabla_\mu L \Big|_{\substack{\theta=\theta^* \\ \mu=\mu^*}} = D_i(y/\theta^*, x) - 1 = 0\,, \tag{9}$$

where $\theta^* = \theta(x,y), \mu^* = \mu(x,y)$ are the optimal solutions to (7). Thus, from (8), we have:

$$\mu^* = -1/[\nabla'_{y/\theta} D_i(y/\theta^*, x) y/(\theta^*)^2]. \qquad (10)$$

And, since θ^* is a solution to optimization problem (7), its value must be equal to unity, and so:

$$\mu^* = -1/\nabla'_y D_i(y,x) y. \qquad (11)$$

The right-hand side in (11) is non-zero due to (4), so $\mu^* \neq 0$, i.e., the constraint in (6) is binding at the optimal values, i.e., $D_i(y,x) = 1$. Thus, due to (5), equation (11) reduces to:

$$\mu^* = 1/e_i(y,x). \qquad (12)$$

On the other hand, note that the envelope theorem applied to (7) tells us that:

$$\nabla'_x D_o(x,y) = \nabla'_x L(\theta^*, \mu^* \mid x, y) = -\mu^* \nabla'_x D_i(y/\theta^*, x). \qquad (13)$$

Post-multiplying both sides of (13) by $-x$ and using again that $\theta^* = 1$, we get:

$$-\nabla'_x D_o(x,y) x = \mu^* \nabla'_x D_i(y,x) x. \qquad (14)$$

The left-hand side in (14) is assumed to be non-zero according to (4) and equals $e_o(x,y)$ according to (5) since $D_o(x,y) = 1$ at the optimum. Because $D_i(y,x)$ is homogeneous of degree one in x, Euler's rule tells us that $\nabla'_x D_i(y,x) x = D_i(y,x)$, which is equal to unity at the optimum. Thus, (14) becomes:

$$e_o(x,y) = \mu^*. \qquad (15)$$

Combining this last result in (14) with (12), we get the desired expression:[3]

$$e_o(x,y) = 1/e_i(y,x).$$

Intuitively, the theorem above tells us that the two scale elasticity formulas measure the same property of technology equivalently and we can get one from the other by just taking the reciprocal, provided that certain condition is satisfied. Importantly, this condition is the necessary and sufficient condition for this equivalence result to hold and it states that, at the points where elasticity is measured, the gradients of the input and output distance functions shall not be orthogonal to the output and input vectors, respectively. In turn, this technical requirement ensures not running into a situation of division by zero (as can be seen among the steps in the proof of the theorem). By this condition, we ensure that at a given point of measurement, neither measure of scale elasticity explodes or degenerates to zero, and then (and only then) they give equivalent information about the scale of technology at that point. On the other hand, the necessary and sufficient condition (4) also has economic meaning: it requires that an increase in all inputs by the same proportion shall cause some proportional non-zero change in all outputs, and, naturally, it should not be an infinite increase. In turn, this condition also implies that, for the equivalence of the two measures, the technology must be so that $D_o(x, y) = 1 \Leftrightarrow D_i(y, x) = 1$ at a point where the elasticity is measured, as is also seen from the steps of the proof of the theorem.

It is also worth noting that a side fruit of the proof of the theorem above is an interesting economic intuition of the scale elasticity measures. Specifically, (15) tells us that the output oriented scale elasticity has Lagrange multiplier meaning—it is the shadow price of relaxing the constraint, in the optimization problem (6). A similar argument can be made about the input oriented scale elasticity measure being the Lagrange multiplier or the shadow price of relaxing the constraint in optimization of the input distance function defined in terms of the constraint on the output distance function, formulated analogous to (6). Such Lagrange multiplier interpretations of scale elasticity measures also give a simple way of obtaining and analyzing a measure of scale elasticity, without taking derivatives as in (1), (2), and (5).

Notes

1. See also Hanoch (1975), Panzar and Willig (1977), Banker (1984), Banker et al. (1984), Färe et al. (1986), Banker and Thrall (1992), Førsund (1996), Golany and Yu (1997), Fukuyama (2000, 2003), Førsund and Hjalmarsson (2004), Krivonozhko et al. (2004), Hadjicostas and Soteriou (2006, 2010), and Podinovski et al. (2009), to mention just a few.
2. It is worth noting that definitions in (1) and (2) relate the vector of inputs to the vector of outputs implicitly, and so we rely on the implicit function theorem and assumptions required by it.
3. Note that a similar proof can be established by starting from the definition of $D_i(y, x)$ in (6).

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