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A Modified ADF Test for Geometric ARMA Processes

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Abstract

This study first shows that geometric processes arise naturally and then proposes a modification to the standard augmented Dickey-Fuller (ADF) equation for testing unit root non-stationarity in geometric ARMA (r,q) processes with $q \geq 1$. The proposed test equation removes the order truncation bias that is inherent in the standard ADF test equation for such processes.

Key words: autoregressive moving average processes; order approximation; infinite order

JEL classification: C12; C22

1. Introduction

The standard augmented Dickey-Fuller (ADF) equation for unit root testing is designed primarily for autoregressive (AR) processes of finite lag order. In the absence of feasible procedures that deal directly with infinite-order AR processes, approximations by finite-order processes have taken a prominent role. The order approximation methods are particularly relevant to autoregressive moving average ARMA (r,q) processes with $q \geq 1$ due to the fact that the invertible MA components of such processes necessarily generate AR representations of infinite lag order (Hamilton, 1994, p. 60). Hence, the standard ADF test equation, which is a finite-order approximation of such infinite-order AR representations, necessarily contains an infinite number of zero parametric restrictions.

A number of studies in the literature have indicated that the omitted variable bias associated with such finite-order approximations are likely contributors to the well-known low power and size distortions associated with the standard ADF procedure (Ng and Perron, 2001; Harvey et al., 2009). Meanwhile, various extensions of the standard ADF test equation have emerged in the literature, including the ESTAR and Fourier extensions of Kapetanios et al. (2003) and Enders and Lee (2012). Those studies arbitrarily extend the ADF test equation in order to

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accommodate certain empirical observations, such as nonlinear mean reversion and structural breaks (Firoozi and Lien, 2016).

This study first demonstrates that a subset of ARMA (r,q) processes with $q \geq 1$ naturally possess a property that we define as the geometric property. Briefly, a process satisfies this property if its AR representation is an infinite-order geometric series. The study then establishes that, within the class of geometric ARMA processes, the standard ADF test equation is a truncated version that needs a specific extension to eliminate its omitted variable bias. It then proposes an alternative test equation that directly accommodates the infinite order of the AR representations without relying on any finite-order truncation or approximation, thus removing the omitted variable bias of the standard ADF test equation for geometric processes.

Section 2 briefly reviews the relevant notions and the finite-order approximations that are inherent in the standard ADF test equations when applied to ARMA (r,q) processes with $q \geq 1$. Section 3 first shows that the geometric pattern arises naturally in the AR(∞) representations of some ARMA (r,q) processes with $q \geq 1$ and then accordingly defines the geometric subset of ARMA processes. Section 4 proposes a unit root test equation for geometric ARMA (r,q) processes. A number of implementation guidelines and connections to the standard ADF test are presented in the last section.

2. Order Approximation in the Standard ADF Equation

The general ARMA (r,q) model can be written (Hamilton, 1994, p. 64):

$$B(L) = C(L)\varepsilon_t \,, \tag{1}$$

where (ε_t) is white noise, L is the lag operator, $B(L) = 1 - \sum_{i=1}^{r} b_i L^i$, and $C(L) = 1 - \sum_{i=1}^{q} c_i L^i$. If $q \geq 1$ and the MA component is invertible, then the roots of the polynomial $C(L)$ are outside the unit circle and the process in (1) has the AR(∞) representation $A(L)y_t = \varepsilon_t$, where:

$$A(L)y_t = \frac{B(L)}{C(L)} = 1 - \sum_{i=1}^{\infty} a_i L^i \ . \tag{2}$$

The convergence of the infinite series $\sum_{i=1}^{\infty} a_i$ follows from the invertibility condition. Throughout this study we consider ARMA (r,q) processes with $q \geq 1$ and assume that the MA components are invertible, thus ARMA processes necessarily have a convergent AR(∞) representation of the form $A(L)y_t = \varepsilon_t$, where $A(L)$ is defined as in (2). Consider then the following two cases.

Case1. (y_t) is non-stationary. In this case $A(L) = (1-L)A^*(L)$, where the roots of $A^*(L)$ lie outside the unit circle, and by the commutativity of $(1-L)$ and $A^*(L)$, (y_t) has the stationary AR(∞) representation $A^*(L)\vartriangle y_t = \varepsilon_t$, which is written equivalently as:

$$\triangle y_t = \sum_{i=1}^{\infty} a_i^* \triangle y_{t-i} + \varepsilon_t \,. \tag{3}$$

Case 2. (y_t) is stationary. In this case $A(L) = 1 - \sum_{i=1}^{\infty} a_i L^i$ with $\sum_{i=1}^{\infty} |a_i| < 1$. Thus (y_t) has the stationary $\mathrm{AR}(\infty)$ representation $A(L) y_t = \varepsilon_t$, which is written equivalently as:

$$y_t = \sum_{i=1}^{\infty} a_i \, y_{t-i} + \varepsilon_t \,. \tag{4}$$

Adding and subtracting $a_j y_{t-j+1}$ repeatedly shows that the $\mathrm{AR}(\infty)$ representation in (4) is equivalent to (Enders, 2010, p. 215, 219):

$$\triangle y_t = \gamma y_{t-1} + \sum_{i=1}^{\infty} \beta_i \triangle y_{t-i} + \varepsilon_t \,. \tag{5}$$

The restriction $\gamma = 0$ reduces the stationary representation (5) to the non-stationary representation (3). A consequence is the following remark.

Remark 1. Testing for a unit root in an $\mathrm{ARMA}(r,q)$ process (y_t) with $q \ge 1$ is equivalent to testing the null hypothesis:

$$H_0 : \gamma = 0 \tag{6}$$

on the infinite-order test equation in (5). Failure to reject H_0 lends support to the presence of a unit root in the process.

Implementing the unit root test summarized in Remark 1 requires a finite-order approximation of the infinite-order model in (5) and its restricted version under $\gamma = 0$. The approximation methods in essence suggest evaluating a finite-order version of (5) that generates the standard augmented Dickey-Fuller (ADF) test equation:

$$\triangle y_t = \gamma y_{t-1} + \sum_{i=1}^{p} \beta_i \triangle y_{t-i} + \varepsilon_t \,. \tag{7}$$

The studies in the approximation literature (Said and Dickey, 1984; Xiao and Phillips, 1998; Ng and Perron, 2001; Chang and Park, 2002; Enders, 2010, p. 219) contain various aspects of identifying a finite truncation lag order (p). Said and Dickey (1984), Xiao and Phillips (1998), and Chang and Park (2002) have shown that the null distribution of the ADF test statistic under the ADF test equation (7) remains asymptotically valid as long as the choice of finite lag order p satisfies a set of boundedness conditions in regard to the sample size T. However, such asymptotic results on the null distribution of the ADF test statistic do not mitigate the fact that the AR representations of $\mathrm{ARMA}(r,q)$ processes with $q \ge 1$ are of infinite order and the standard ADF test equation applied to such processes necessarily imposes an infinite number of zero parametric restrictions. Within a subset of $\mathrm{ARMA}(r,q)$ processes with $q \ge 1$ as defined in the next section, we present a finite-order test equation that directly accommodates the infinite order of AR representations without imposing any parametric restriction.

3. Geometric ARMA Processes

In this section we first show that the geometric pattern arises naturally in a subset of AR representations of ARMA (r,q) processes with $q \geq 1$ and then define the class of geometric processes based on this pattern. The geometric structure arises in the classic models of adaptive expectation and has a history of application in economic settings (Theil, 1971, p. 262). The existence and prevalence of geometric pattern in the AR representations of ARMA (r,q) processes with $q \geq 1$ is shown in the following cases. First, for the ARMA(0,1) process $y_t = (1-\lambda L)\varepsilon_t$ with the invertibility condition $|\lambda| < 1$, the geometric pattern arises in the parameters of its $\mathrm{AR}(\infty)$ representation:

$$y_t = -\sum_{i=1}^{\infty} \lambda^i y_{t-i} + \varepsilon_t .$$

For the more general invertible ARMA $(0,q)$ process, the coefficients of its $\mathrm{AR}(\infty)$ representation emerge from the product of q geometric infinite series. The same product is also present in $\mathrm{AR}(\infty)$ representations of general invertible ARMA (r,q) processes with $q \geq 1$. Specifically, consider the ARMA process $B(L)y_t = C(L)\varepsilon_t$ with $q \geq 1$, where $(1/\lambda_n)_{n=1}^{q}$ are the roots of the polynomial $C(L)$. Its $\mathrm{AR}(\infty)$ representation is then defined by $A(L)y_t = \varepsilon_t$, where:

$$A(L) = \frac{B(L)}{C(L)} = B(L)\prod_{n=1}^{q}(1-\lambda_n L)^{-1} .$$

It follows from the invertibility condition $|\lambda_n| < 1$ for $n = 1, 2, \ldots, q$ that each component $(1-\lambda_n)^{-1} = \sum_{j=0}^{\infty} \lambda_n^j$ is a convergent geometric series, hence $(1-\lambda_n L)^{-1} = \sum_{j=0}^{\infty} \lambda_n^j L^j$. In the case $q = 2$, assuming that the roots $|\lambda_1| < 1$ and $|\lambda_2| < 1$ satisfy $|\lambda_1 + \lambda_2 - \lambda_1\lambda_2| < 1$, the product $\Pi_{n=1}^{2}(1-\lambda_n)^{-1}$ is equivalent to the convergent geometric series $\sum_{j=0}^{\infty}(\lambda_1 + \lambda_2 - \lambda_1\lambda_2)^j$ as shown by:

$$\Pi_{n=1}^{2}(1-\lambda_n)^{-1} = \frac{1}{1-(\lambda_1 + \lambda_2 - \lambda_1\lambda_2)} = \sum_{j=0}^{\infty}(\lambda_1 + \lambda_2 - \lambda_1\lambda_2)^j .$$

The following definition is based on the stated cases.

Definition. An invertible ARMA (r,q) process (y_t) with $q \geq 1$ is defined to be a geometric type if the parameters (β_i) of its $\mathrm{AR}(\infty)$ representation in (5) satisfy the following pattern of geometric decay:

$$\beta_i = \pi\lambda^i \text{, for some } \pi \text{ and } \lambda \text{ with } 0 < |\lambda| < 1. \tag{8}$$

Substitution of (8) into (5) for (β_i) shows that a value of $|\lambda|$ closer to zero results in a faster convergence by giving a smaller weight to distant changes in (y_t) relative to recent changes in (y_t). Since the stationary process in (5) reduces to the non-stationary process in (3) by setting $\gamma = 0$, a geometric process may be stationary or non-stationary.

4. Modified ADF Test for Geometric Processes

In the remainder of this study, we focus on geometric ARMA processes as defined above. We show that the geometric property allows us to devise an equivalent finite-order unit root test equation that directly accommodates the infinite order in the AR representation in (5) without any need for finite-order truncations inherent in the standard ADF test equation shown in (7). In the presence of the geometric pattern defined in (8), the infinite-order test equation in (5) is written equivalently in the form:

$$\triangle y_t = \gamma y_{t-1} + \pi \sum_{i=1}^{\infty} \lambda^i \triangle y_{t-i} + \varepsilon_t \, . \tag{9}$$

Multiplying the one lag version of (9) by λ gives:

$$\lambda \triangle y_{t-1} = \lambda\gamma y_{t-2} + \pi \sum_{i=2}^{\infty} \lambda^i \triangle y_{t-i} + \varepsilon_{t-1} \, . \tag{10}$$

Subtracting (10) from (9) then leads to:

$$\begin{aligned} \triangle y_t &= \gamma y_{t-1} - (\lambda\gamma) y_{t-2} + (\lambda + \lambda\pi) \triangle y_{t-1} + (\varepsilon_t - \lambda\varepsilon_{t-1}) \\ &= (\gamma + \lambda + \lambda\pi) y_{t-1} + (-\lambda\gamma - \lambda - \lambda\pi) y_{t-2} + (\varepsilon_t - \lambda\varepsilon_{t-1}). \end{aligned} \tag{11}$$

Define:

$$\delta_t = \varepsilon_t - \lambda\varepsilon_{t-1} \, , \tag{12}$$

$$\phi_1 = \gamma + \lambda + \lambda\pi \, , \tag{13}$$

$$\phi_2 = -(\lambda\gamma + \lambda + \lambda\pi) \, . \tag{14}$$

Then (11) is written as:

$$\triangle y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \delta_t \, . \tag{15}$$

Under the geometric pattern in (8) and the definitions in (12)–(14), the finite-order equation in (15) is not an approximation or truncation but equivalent to the infinite-order AR representation in (5). The following theorem is then a consequence of Remark 1 and the definitions in (12)–(14).

Theorem 1. The un-truncated ADF testing for a unit root in a geometric ARM (r,q) process (y_t) with $q \geq 1$ is equivalent to testing the hypothesis:

$$H_0^{'} : \phi_1 + \phi_2 = 0 \tag{16}$$

on the finite-order test equation (15). Failure to reject $H_0^{'}$ lends support to the presence of a unit root in the process.

Proof. Remark 1 showed that the un-truncated ADF unit root test is equivalent to testing the hypothesis $\gamma = 0$ on the infinite-order test equation (5) with the standard finite-order approximation shown in (7). When the process is geometric as defined by (8) and under the definitions in (12)–(14), the steps in (9)–(15) showed that the un-truncated infinite-order test equation (5) reduces to the finite-order equation (15) without use of any truncation or approximation. Further, it follows from the definitions in (13)–(14) and the geometric condition $0 < |\lambda| < 1$ in (8) that the unit root hypothesis $\gamma = 0$ is equivalent to the hypothesis $\phi_1 + \phi_2 = 0$.

Implementation of the test proposed by Theorem 1 requires estimation of the test equation (15). The error terms δ_t in (15) are autocorrelated as shown by the definition in (12):

$$\gamma_1 = \text{cov}(\delta_t, \delta_{t-1}) = E[(\varepsilon_t - \lambda\varepsilon_{t-1})(\varepsilon_{t-1} - \lambda\varepsilon_{t-2})] = -\lambda\sigma^2 , \tag{17}$$

where $\text{var}(\varepsilon_t) = \sigma^2$. Error autocorrelations cause the ordinary least squares estimators to be inefficient, but the issue is resolved by a number of standard estimation methods for regressions with autocorrelated error terms, such as generalized least squares, autocorrelation-consistent variance estimation, and generalized method of moments (Greene, 2008).

5. Concluding Remarks

This study has shown that the geometric processes as defined above arise naturally as a subset of ARMA (r, q) processes with $q \geq 1$. We have established that, within the class of geometric processes, the standard ADF test equation is a truncation of the true infinite-order equation and, hence, carries an omitted variable bias. When the underlying process is geometric, the test proposed in Theorem 1 is superior to the standard ADF test in the sense that it removes the omitted variable bias of the standard ADF test. As shown earlier, geometric processes have reasonable structures and the investigator may have prior belief that the underlying generator is a geometric process. In such cases, the testing approach proposed in Theorem 1 has an unbiasedness advantage relative to the standard ADF test. In the absence of a prior belief that the underlying process is geometric, the difficulty is that currently there is no known test that characterizes an empirical process as being generated by a geometric process. However, some guidance is provided next.

The standard ADF test equation is the one stated in (7) with the finite lag order p . The standard literature on ADF testing contains methods that identify the value of p for a given set of observations on a process (y_t) . As is shown next, the possibility that the true generating process is a geometric process arises when the order determination step in the standard ADF procedure results in the value $p = 1$. In case $p = 1$, the standard ADF test equation (7) reduces in form to the test equation (15) proposed in Theorem 1 as shown by:

$$\begin{aligned} \triangle y_t &= \gamma y_{t-1} + \beta \triangle y_{t-1} + \varepsilon_t = (\gamma+\beta) y_{t-1} + (-\beta) y_{t-2} + \varepsilon_t \\ &= \phi_1' y_{t-1} + \phi_2' y_{t-2} + \varepsilon_t . \end{aligned} \tag{18}$$

The determination that $p=1$ suggests that both γ and β in the test equation (18) are non-zero. It then follows from the definitions of ϕ_1' and ϕ_2' given in (18) that the standard ADF unit root hypothesis $\gamma=0$ is equivalent to the hypothesis $\phi_1' + \phi_2' = 0$, which is the analog of the hypothesis (16) proposed in Theorem 1. It follows that the standard ADF test in cases where the lag order determination step establishes the lag order value $p=1$ is an analog of the test proposed in Theorem 1 for geometric ARMA processes in the sense that the two tests have identical test equation forms and identical unit root hypotheses. The only difference is that, because of the error autocorrelation established in (17) for geometric processes, the test proposed in Theorem 1 requires more general estimation techniques than ordinary least squares. Hence, in cases where the lag order determination step of the standard ADF test results in the lag order value $p=1$, the set of possible generating processes includes the geometric processes, thus it is advisable to apply the general estimation methods that accommodate error autocorrelations for the computation of its test statistic.

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