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On Farrell's Decomposition and Aggregation

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Abstract

In this paper we show that in order to aggregate individual efficiency scores into a group (e.g., industry) efficiency score in such a way that the multiplicative structure of further decompositions is preserved with equal weights across components, the weighted geometric mean is required. We also show how the weights can be chosen using a variation of a theorem by Koopmans (1957). In the end, our paper provides a mathematically consistent and theoretically justified way of aggregation of Farrell-type efficiency scores.

Key words: Farrell efficiency; index aggregation

JEL classification: D24

1. Introduction

In the classic paper "The Measurement of Productive Efficiency," Farrell (1957) furnishes a decomposition of his cost efficiency index, or overall efficiency as he terms it. He shows that this index can be multiplicatively partitioned into a technical component and an allocative component, which he calls price efficiency. Farrell also discusses the aggregation of firm efficiency into industry efficiency measures, but he did not discuss under what conditions such aggregation could be performed while preserving the decomposition.

In this paper we show that in order to aggregate individual efficiency into group (e.g., industry) efficiency in such a way that the multiplicative structure of further decompositions is preserved with equal weights across components, the weighted geometric mean is required. We also show how the weights can be chosen using a

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variation of Koopmans's (1957) aggregation theorem. Thus, our paper provides a researcher with a mathematically consistent and theoretically justified way of aggregating Farrell-type efficiency indexes.

2. The Model

In this paper we derive results for Farrell cost efficiency, but of course our methodology applies to other cases (e.g., revenue efficiency) as well.

Let r_k and s_k ($k=1,2$) be firm k's two component measures of efficiency and let their product $q_k = r_k s_k$ be the Farrell cost measure of overall efficiency. In general $k=1,\ldots,K$, but for the sake of notational simplicity, we present the results for the case when $k=1,2$. Suppose we want to aggregate these measures into group (e.g., industry of K firms) measures while preserving the multiplicative structure of decomposition. This results in the following functional equation:

$$V(q_1, q_2) = V(r_1, r_2)\ V(s_1, s_2)\,, \tag{1}$$

where V is some function that aggregates individual indices (over all k). Let us generalize this equation by introducing a set of parameters $z=(z_1,\ldots,z_J)\in\Re^J$, i.e.:

$$U(q_1, q_2; z) = U(r_1, r_2; z)\ U(s_1, s_2; z)\,. \tag{2}$$

The solution to this equation is (Aczél, 1990, p. 27; Eichhorn, 1978, p. 94):

$$U(q_1, q_2; z) = q_1^{\alpha_1(z)} q_2^{\alpha_2(z)}\,, \tag{3.a}$$

$$U(r_1, r_2; z) = r_1^{\alpha_1(z)} r_2^{\alpha_2(z)}\,, \tag{3.b}$$

$$U(s_1, s_2; z) = s_1^{\alpha_1(z)} s_2^{\alpha_2(z)}\,, \tag{3.c}$$

where $\alpha_k(z)$ are arbitrary functions of z. Thus, we have shown that aggregating the cost efficiency while preserving the decomposition and equal weights across components, requires a weighted geometric mean procedure, i.e.:

$$\begin{aligned}\left(q_1^{\alpha_1(z)} q_2^{\alpha_2(z)}\right) &= \left(r_1^{\alpha_1(z)} s_1^{\alpha_1(z)}\right)\times\left(r_2^{\alpha_2(z)} s_2^{\alpha_2(z)}\right)\\ &= \left(r_1^{\alpha_1(z)} r_2^{\alpha_2(z)}\right)\times\left(s_1^{\alpha_1(z)} s_2^{\alpha_2(z)}\right),\end{aligned} \tag{4}$$

where the group indexes are $\left(q_1^{\alpha_1(z)} q_2^{\alpha_2(z)}\right)$, $\left(r_1^{\alpha_1(z)} r_2^{\alpha_2(z)}\right)$, and $\left(s_1^{\alpha_1(z)} s_2^{\alpha_2(z)}\right)$.

Next, for practical purposes, we want to determine the weights $\alpha_1(z)$ and $\alpha_2(z)$. For this, define the input requirement set of the group as:

$$\overline{L}(y^1, y^2) = L^1(y^1) + L^2(y^2)\,, \tag{5}$$

where y^1 and y^2 are output vectors for each firm and $L^1(y^1)$ and $L^2(y^2)$ are the firms' input requirement sets, i.e.:

$$L^k(y^k) = \{x^k \in \Re^N_+ : x^k \text{ can produce } y^k\}, \quad y^k \in \Re^M_+,\ k = 1, 2\,. \tag{6}$$

Given a vector of input prices $w \in \Re^N_+$ equal across all k, the group and the firm's cost functions are given by:

$$\overline{C}(y^1, y^2, w) = \min_x \{wx : x \in \overline{L}(y^1, y^2)\} \tag{7}$$

and

$$C^k(y^k, w) = \min_{x^k} \{wx^k : x^k \in L^k(y^k)\}, \tag{8}$$

respectively.

The following statement is a variation of Koopmans' (1957) aggregation theorem:

$$\overline{C}(y^1, y^2, w) = C^1(y^1, w) + C^2(y^2, w)\,. \tag{9}$$

Proof of this statement can be found in Färe et al. (2004) and for the revenue analogue in Färe and Zelenyuk (2003). From the last expression it follows that the group cost efficiency index is the share-weighted average of the efficiencies of all the firms within the group, i.e.:

$$Q \equiv \frac{\overline{C}(y^1, y^2, w)}{\sum_{n=1}^{N} w_n(x_{1n} + x_{2n})} = \frac{C^1(y^1, w)}{\sum_{n=1}^{N} w_n x_{1n}} S^1 + \frac{C^2(y^2, w)}{\sum_{n=1}^{N} w_n x_{2n}} S^2, \tag{10}$$

where the weights are the firm's cost shares, i.e.:

$$S^k = \frac{\sum_{n=1}^{N} w_n x_{kn}}{\sum_{n=1}^{N} w_n(x_{1n} + x_{2n})}, \quad k = 1, 2\,. \tag{11}$$

It might be worth noting that an advantage of these weights is that they are not ad hoc (although might be exactly what one would guess them to be) but derived from economic optimization behavior under certain assumptions. Besides usual regularity conditions on technology, the critical assumptions include the additive aggregation structure (5) that we imposed on the group technology and the law of one price on all input markets (all firms face the same input prices). As a result, the group cost efficiency (or group overall efficiency) index would be:

$$Q = q_1 S^1 + q_2 S^2 . \tag{12}$$

Next, taking the first-order Taylor series approximation of (3.a) around $q_1 = q_2 = 1$ (which is a natural point for the Farrell-type efficiency index to be approximated around), we obtain:

$$U(q_1, q_2; z) \cong 1^{\alpha_1(z)} 1^{\alpha_2(z)} + \alpha_1(z)\left(1^{\alpha_1(z)-1} 1^{\alpha_2(z)}\right) + \alpha_2(z)\left(1^{\alpha_1(z)} 1^{\alpha_2(z)-1}\right),$$

i.e.,

$$U(q_1, q_2; z) \cong \alpha_1(z) q_1 + \alpha_2(z) q_2 . \tag{13}$$

By equating (12) and (13) we get:

$$\alpha_1(z) = S^1 \quad \text{and} \quad \alpha_2(z) = S^2 ,$$

which gives us particular weights that can be used for the geometric aggregation suggested in (3.a)-(3.c).

3. Concluding Remarks

In this paper we present a practical way to aggregate the overall Farrell efficiencies of individual firms into the group (e.g., industry) efficiency index so that the decomposition that exists on the disaggregated level is also preserved on the aggregate level with equal weights across components. Such aggregation is based on the weighted geometric mean. To determine economically meaningful weights we turn to a cost function analogue of the Koopmans (1957) theorem for aggregation of profit functions and obtain as aggregation weights the observed cost shares of individual firms in the group. This approach should prove to be useful for researchers challenged with a question of efficiency of industries as well as various groups (e.g., regulated vs. non-regulated, foreign vs. domestic, etc.) within such industries.

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