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Does Asymmetric Dependence Structure Matter? A Value-at-Risk View

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Abstract

To investigate the importance of asymmetric dependence structures for portfolio value-at-risk (VaR) and conditional VaR (CVaR) calculations, we introduce bivariate copula functions with two GJR-GARCH models as marginals. The results show that the copula models and the competing dynamic conditional correlation (DCC) model are valid for almost all two-asset portfolios with different weights. However, among models validated with standard procedures, copula models with asymmetric dependence structures can save capital charges for market risks and reduce potential loss compared with those with symmetric dependence structures and with the competing DCC model, implying that asymmetric dependence structures are of great importance in improving VaR and CVaR calculations not only from a statistical but also an economic perspective.

Key words: value-at-risk; asymmetry; dependence structure; copula; multivariate GARCH model

JEL classification: C22; G32

1. Introduction

Value-at-risk (VaR), the maximum expected loss in market value of a position or a portfolio of assets over a given time horizon with a small probability, has been widely applied in risk management. When actual losses exceed VaRs calculated by a given model, VaRs cannot provide an improved measure of potential risk. Another risk measure, conditional VaR (CVaR), the expected value of actual losses exceeding VaRs, becomes important in this context. Most conventional literature measures portfolio VaR via a simplification using an analytical method (also known as the variance-covariance method) that assumes multivariate normality (e.g., Billio and Pelizzon, 2000; Palaro and Hotta, 2006). However, fat-tailedness and/or an asymmetric dependence structure (i.e., co-movement patterns), which have been observed in joint distributions of financial asset returns, make the analytical method

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impractical. Although the use of t-distributed errors and the GARCH model can mitigate the effects of fat-tailedness, the problem of an asymmetric dependence structure remains. Giot and Laurent (2003) argued for using an asymmetric distribution assumption to improve VaR measurements.

To incorporate a general class of dependence structures, this paper introduces the copula approach proposed by Sklar (1959) into VaR estimation. The copula approach is one of the most important developments in modeling dependence structures in multivariate distributions and has become popular in the statistical literature (e.g., Genest and Rivest, 1993; Nelsen, 1999). Mendes (2005), Hu (2006), Patton (2006), and Lai et al. (2009) applied the copula method to empirical studies in stock markets, exchange rates, and hedging strategies of stock spot and futures. This approach states that any k-dimensional joint distribution function may be decomposed into k marginal distributions and a copula function that completely describes the dependence among the k variables. Since the marginals and the joint distribution constructed via a copula do not necessarily belong to the same family, this approach introduces flexibility in modeling joint distributions beyond the typical multivariate normal or multivariate t distributions. In addition, using the inference for the margins estimation method proposed by Joe and Xu (1996), the parameters of the marginal distributions and the copula function can be estimated separately, and thus estimation becomes easy.

Given the feasibility of incorporating an asymmetric dependence structure in a VaR model, it is still important to ask whether a model with an asymmetric dependence structure outperforms one with a symmetric dependence structure. According to prevailing statistical wisdom, model performance may be assessed using the so-called backtesting procedure suggested by The Basel Committee (1996). This procedure specifies that a VaR model performs well when there are a probabilistically reasonable number of exceptions (i.e., actual losses exceed VaRs). However, if many models perform well based on the backtesting procedure with similar statistical accuracy, it is unclear how further model selection should proceed, a real problem for practitioners. Banks, financial institutions, and non-financial corporations in practice can choose an overall risk level they are willing and able to take according to VaRs and then authorize the risk limits to the subsidiaries or departments to pursue the greatest risk-adjusted returns. Moreover, financial institutions have been required by regulators to set the capital requirement for market risk of the holding portfolio suggested by a valid internal VaR model, which is has come to be known as the internal models approach since the 1996 Amendment of the Basel Accord (1996). Among all valid models, a conservative VaR model which has fewer exceptions (i.e., actual losses that exceed the VaRs) and larger absolute values of VaRs could result in inefficiency in using capital and thereby raise opportunity costs. In contrast, an active VaR model may reduce capital requirements bearing no interest such that the corporation may invest more capital in profitable projects. For example, if a bank with USD 50 million capital requirements per year could save 6% capital charges and invest it at a 5% rate of return, it may make USD 50 × 0.06 × 1.05 = 3.15 million extra profits. An active VaR model of course may also cause greater losses and exposures than does a conservative one. But if both the conservative and active VaR models are probabilistically acceptable and the goal of the corporation is to take risk in order to increase the equity value of shareholders, the corporation has an incentive to undertake a riskier project (Galai and Masulis, 1976) such as the active VaR model. From this viewpoint and economic intuition, a good VaR model could be said to pass the backtesting procedure and have less capital charges and lower potential losses. Accordingly, this study evaluates valid VaR model performance via capital saving (the percentage difference of VaR) and potential loss reduction (the percentage difference of CVaR).

This paper applies the GJR-GARCH model (Glosten et al., 1993) with seven copula functions (copula-GJR-GARCH) to calculate VaRs and CVaRs of two-asset portfolios constructed from three major stock indices. Both long and short trading positions are considered. Five of the copula-GJR-GARCH models have the same acceptability as the dynamic conditional correlation (DCC) model of Engle (2002) in long positions and two in both long and short positions. Moreover, based on two criteria of relative performance defined by capital saving and expected loss reduction, four of the copula-GJR-GARCH models with asymmetric dependence structures outperform the symmetric DCC and the symmetric Gaussian copula model.

The remainder of the paper is organized as follows. Section 2 discusses the methodology for the VaR and the validation criteria. Section 3 provides the data description, presents the empirical results, and evaluates performance across the different models. Section 4 concludes.

2. VaR Computation: Copula Simulation Methodology

2.1 Copula Function

A copula function is a joint distribution function of a vector of uniform random variables. According to Sklar's theorem, since any vector of random variables can be mapped into a vector of uniform random variables by the corresponding distribution functions, the joint distribution can be represented in terms of a copula function. Therefore, any joint distribution function F of k return series $R_{1,t},\ldots,R_{k,t}$ can be decomposed into k marginal distributions $F_1,\ldots,F_k$ and a copula function C that completely describes the dependence structure among the k return series:

$$F\left(R_{1,t},\ldots,R_{k,t};\theta_1,\ldots,\theta_k,\theta_c\right)=C\left(F_1(R_{1,t};\theta_1),\ldots,F_k(R_{k,t};\theta_k);\theta_c\right), \tag{1}$$

where $\theta_1,\ldots,\theta_k$ are parameter sets for marginal distributions $F_1,\ldots,F_k$ respectively and θ_c is the parameter set for the copula function.

Assuming all cumulative distribution functions are differentiable, the bivariate joint density is given by:

$$f\left(R_{1,t},R_{2,t};\theta_1,\theta_2,\theta_c\right)=\frac{\partial^2 C\left(F_1(R_{1,t};\theta_1),F_2(R_{2,t};\theta_2);\theta_c\right)}{\partial R_{1,t}\partial R_{2,t}}$$
$$=c\left(F_1(R_{1,t};\theta_1),F_2(R_{2,t};\theta_2);\theta_c\right)\cdot\prod_{k=1}^{2}f_k\left(R_{k,t};\theta_k\right), \quad (2)$$

where $c(u_{1,t},u_{2,t};\theta_c)=\partial C(u_{1,t},u_{2,t};\theta_c)/\partial u_{1,t}\partial u_{2,t}$ is the copula density function and $u_{1,t}=F_1(R_{1,t};\theta_1)$ and $u_{2,t}=F_2(R_{2,t};\theta_2)$. Thus, the bivariate density function of $R_{1,t}$ and $R_{2,t}$ can be written as the product of the copula density and the two marginal densities $f_1(R_{1,t};\theta_1)$ and $f_2(R_{2,t};\theta_2)$. The log-likelihood function is then:

$$\log f\left(R_{1,t},R_{2,t};\theta_1,\theta_2,\theta_c\right)$$
$$=\log c\left(F_1(R_{1,t};\theta_1),F_2(R_{2,t};\theta_2);\theta_c\right)+\sum_{k=1}^{2}\log f_k\left(R_{k,t};\theta_k\right). \quad (3)$$

To simplify notation, we represent (3) as:

$$L(\Theta)=L_c(\theta_c)+\sum_{k=1}^{2}L_k(\theta_k), \quad (4)$$

where $\Theta=\{\theta_1,\theta_2,\theta_c\}$ and $L_c(\cdot)$ and $L_k(\cdot)$ are the log-likelihood functions of the copula and $R_{k,t}$.

It is sometimes difficult to achieve optimization when the number of model parameters of the bivariate density is large. Joe and Xu (1996) proposed a two-step estimation called inference for the margins (IFM) to overcome estimation challenges. By this method, one can estimate the marginal densities and the copula density separately. We follow this approach in this paper.

In the first step of the IFM method, the parameters of the marginal distributions are estimated via maximum likelihood as:

$$\hat{\theta}_1=\arg\max\sum_{t=1}^{T}\log f_1(R_{1,t};\theta_1), \quad (5)$$

$$\hat{\theta}_2=\arg\max\sum_{t=1}^{T}\log f_2(R_{2,t};\theta_2). \quad (6)$$

In the second step, the marginal CDFs with the first-step estimates are applied to $R_{1,t}$ and $R_{2,t}$ to provide estimates of the probabilities $u_{1,t}$ and $u_{2,t}$, which are then used to estimate the copula parameters via:

$$\hat{\theta}_c=\arg\max\sum_{t=1}^{T}\log c(\hat{u}_{1,t},\hat{u}_{2,t};\theta_c). \quad (7)$$

Joe (1997) shows the efficiency of the easily implemented IFM method compared with the usual ML method. The software package RATS of Estima® and the BHHH numerical optimization algorithm are used to obtain estimates and standard errors. No restriction is imposed during estimation.

2.2 Marginal Distributions in a GJR-GARCH Framework

There are two key components in the bivariate copula representation: two marginal distributions of asset returns and one copula function. To capture the pattern of fat tails and the asymmetric response of past news on asset returns, we specify the marginal densities for asset returns as the widely used GJR-GARCH framework (Engle and Ng, 1993; Kim and Kon, 1994; Fornari and Mele, 1995) with *t*-distributed errors defined by:

$$R_{1,t} = a_0 + a_1 R_{1,t-1} + a_2 R_{2,t-1} + \varepsilon_{1,t} ,$$
$$h_{1,t} = \alpha_0 + \alpha_1 \varepsilon_{1,t-1}^2 + \alpha_2 \varepsilon_{1,t-1}^2 I(\varepsilon_{1,t-1} < 0) + \alpha_3 h_{1,t-1} ,$$
$$\eta_{1,t} = \varepsilon_{1,t} \Big/ \sqrt{h_{1,t}} , \quad \eta_{1,t} \mid \Omega_{t-1} \sim t(v_1) ,$$

and

$$R_{2,t} = b_0 + b_1 R_{1,t-1} + b_2 R_{2,t-1} + \varepsilon_{2,t} ,$$
$$h_{2,t} = \beta_0 + \beta_1 \varepsilon_{2,t-1}^2 + \beta_2 \varepsilon_{2,t-1}^2 I(\varepsilon_{2,t-1} < 0) + \beta_3 h_{2,t-1} ,$$
$$\eta_{2,t} = \varepsilon_{2,t} \Big/ \sqrt{h_{2,t}} , \quad \eta_{2,t} \mid \Omega_{t-1} \sim t(v_2) ,$$

where $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are error terms, $h_{1,t}$ and $h_{2,t}$ are the conditional variances for asset returns, $I(\cdot)$ is the indicator function taking the value 1 when there is a negative shock at time $t-1$ and 0 otherwise, $\eta_{1,t}$ and $\eta_{2,t}$ are standardized residuals following *t* distributions with degrees of freedom v_1 and v_2, and Ω_{t-1} is the past information set. Since both marginal models must condition on the same information set up to $t-1$, we include both lagged return series in each mean equation. The conditional variance processes $h_{1,t}$ and $h_{2,t}$ are allowed to have asymmetric responses to negative or positive previous random shocks.

For the competing DCC model, the two-step estimation method is also employed such that we can forecast the covariance matrix for VaR calculations. However, the two-step estimation of DCC model assumes normality, while our marginal models in the first step are *t* distributions. Therefore, we use the CDF of the *t* distribution with estimated degrees of freedom ($\hat{v}_1$ and $\hat{v}_2$) to transform the original standardized residuals ($\hat{\eta}_{1,t}$ and $\hat{\eta}_{2,t}$) into cumulative probabilities and then map these probabilities using the inverse normal CDF function into standard normal variables $\hat{\eta}_{1,t}^n$ and $\hat{\eta}_{2,t}^n$. The resulting series ($\hat{\eta}_{1,t}^n$ and $\hat{\eta}_{2,t}^n$) will be standard normal if the model fits the data and used to estimate the parameters θ_1 and θ_2 in the second step of estimating the DCC model:

$$\mathbf{Q}_t = \bar{\rho}(1-\theta_1-\theta_2) + \theta_1 \boldsymbol{\xi}_{t-1} \boldsymbol{\xi}_{t-1}^T + \theta_2 \mathbf{Q}_{t-1} , \tag{8}$$

where $\mathbf{Q}_t$ is the covariance matrix of the first-step standardized residuals, $\bar{\rho}$ is the unconditional correlation coefficient, and ξ_t is the vector containing $\hat{\eta}_{1,t}^{n}$ and $\hat{\eta}_{2,t}^{n}$.

2.3 Bivariate Copula Density

The first-step marginal GJR-GARCH parameter estimates provide estimated values of $\hat{u}_{1,t} = F_1(\hat{\eta}_{1,t} \mid \Omega_{t-1})$ and $\hat{u}_{2,t} = F_2(\hat{\eta}_{2,t} \mid \Omega_{t-1})$. These values are then used in the second-step estimation of the copula function, describing the dependence structure between $R_{1,t}$ and $R_{2,t}$.

Since the copula function dominates the dependence structure, the selection of a copula density function for estimation should depend on the characteristics of dependence observed in the data set. To give a comprehensive comparison, we choose two elliptical copula functions (Gaussian and *t*), three Archimedean copula functions (Clayton, Gumbel, and Frank), and two mixture copula functions (the mixture of Clayton and its survival copula and the mixture of Gumbel and its survival copula). Note that the parameters in copula functions are not specified to be time-varying; this allows us to concentrate on differences across dependence structures. However, allowing time-varying dependence can be a good choice when forecasting is the major concern, as in Patton (2004) and Lai et al. (2009). The following subsections summarize the copula densities used in the IFM estimation.

2.3.1 Gaussian Copula

Let $\Phi_{\mathbf{R}}$ be the bivariate standard normal distribution with symmetric, positive definite correlation matrix $\mathbf{R}$. The distribution function of the bivariate Gaussian copula is defined as follows:

$$C_{\mathbf{R}}^{Gaussian}(u_1, u_2) = \Phi_{\mathbf{R}}\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right),$$

where Φ^{-1} is the inverse of the univariate standard normal distribution function. Let $u_i = \Phi(x_i)$, so that $x_i = \Phi^{-1}(u_i)$. The density function of the Gaussian copula can be written as follows:

$$c_{\mathbf{R}}^{Gaussian}(u_1, u_2) = \frac{1}{|\mathbf{R}|^{1/2}} \exp\left(-\frac{1}{2}\varsigma^T\left(\mathbf{R}^{-1} - \mathbf{I}\right)\varsigma\right), \quad (9)$$

where $\varsigma = \left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right)^T$.

2.3.2 *t* Copula

Let $T_{\mathbf{R},v}$ be the bivariate standardized *t* distribution with symmetric, positive definite correlation matrix $\mathbf{R}$ and degrees of freedom v. The distribution function of bivariate *t* copula is defined as follows:

$$C_{\mathbf{R},v}^{T}(u_1, u_2) = T_{\mathbf{R},v}\left(t_v^{-1}(u_1), t_v^{-1}(u_2)\right),$$

where t_v^{-1} is the inverse of the univariate *t* distribution function with degrees of freedom v. Accordingly, the *t* copula density is:

$$c_{\mathbf{R},v}^{T}(u_1,u_2)=|\mathbf{R}|^{-1/2}\frac{\Gamma\left(\frac{v+2}{2}\right)}{\Gamma\left(\frac{v}{2}\right)}\left(\frac{\Gamma\left(\frac{v}{2}\right)}{\Gamma\left(\frac{v+1}{2}\right)}\right)^2\frac{\left(1+\frac{1}{v}\varsigma^T\mathbf{R}^{-1}\varsigma\right)^{-(v+2)/2}}{\prod_{i=1}^{2}\left(1+\frac{\varsigma_i^2}{v}\right)^{-(v+1)/2}}, \tag{10}$$

where $\varsigma_i = t_v^{-1}(u_i)$.

2.3.3 Clayton Copula

The Clayton copula is an asymmetric copula belonging to the Archimedean family. The degree of dependence is higher in the lower tail than in the upper. Genest and MacKay (1986) defined the Archimedean family as:

$$C^{Arch}(u_1,u_2)=\varphi^{[-1]}(\varphi(u_1)+\varphi(u_2)), \tag{11}$$

where φ is a strict generator function which is continuous, decreasing (i.e., $\varphi' < 0$), and convex (i.e., $\varphi'' > 0$) such that $\varphi(1) = 0$ and $\varphi(0) = \infty$. $\varphi^{[-1]}$ is the pseudo-inverse of φ with Dom $\varphi^{[-1]} = [0, \infty]$ and Ran $\varphi^{[-1]} = \mathbf{I}$ given by:

$$\varphi^{[-1]}(z)=\begin{cases}\varphi^{-1}(z) & 0\le z\le\varphi(0)\\ 0 & \varphi(0)\le z\le+\infty.\end{cases} \tag{12}$$

Strict generator functions of an Archimedean copula can be derived from the inverse of the Laplace transform of a given distribution function. Different generator functions will induce different Archimedean copulas.

Given the distribution function and generator function of an Archimedean copula, the density copula can be derived using either of the following relationships:

$$c^{Arch}(u_1,u_2)=\frac{-\varphi''(C^{Arch}(u_1,u_2))\varphi'(u_1)\varphi'(u_2)}{[\varphi'(C^{Arch}(u_1,u_2))]^3}, \tag{13}$$

$$c^{Arch}(u_1,u_2)=\frac{\partial C^{Arch}(u_1,u_2)}{\partial u_1\partial u_2}. \tag{14}$$

The generator function of a Clayton copula is $\varphi(t) = (t^{-\alpha} - 1)/\alpha$, where α is the shape parameter and $\alpha \in [-1,0) \cup (0,+\infty]$. It is easy to see $\varphi(0) = \infty$ and $\varphi(1) = 0$. Using (11) and (12), the distribution function of a Clayton copula can be given by:

$$C_\alpha^{Clayton}(u_1,u_2)=\max\left((u_1^{-\alpha}+u_2^{-\alpha}-1)^{-1/\alpha},0\right),$$

and using (13) or (14), the density function can be expressed as:

$$c_{\alpha}^{Clayton}(u_1,u_2)=\frac{(1+\alpha)(u_1^{-\alpha}+u_2^{-\alpha}-1)^{-(1+2\alpha)/\alpha}}{(u_1u_2)^{1+\alpha}}. \quad (15)$$

2.3.4 Gumbel Copula

The Gumbel copula is also asymmetric. It has a higher degree of dependence in the upper tail. The generator function of the Gumbel copula is $\varphi(t)=(-\log t)^{\alpha}$, $\alpha\in[1,+\infty)$. It is easy to see $\varphi(0)=\infty$ and $\varphi(1)=0$. Using (11) and (12), the distribution function of the Gumbel copula is derived as:

$$C_{\alpha}^{Gumbel}(u_1,u_2)=\exp\left(-\left((-\log u_1)^{\alpha}+(-\log u_2)^{\alpha}\right)^{1/\alpha}\right),$$

and the density function is:

$$c_{\alpha}^{Gumbel}(u_1,u_2)=\frac{C_{\alpha}^{Gumbel}(u_1,u_2)(\log u_1\log u_2)^{\alpha-1}\left(-\log C_{\alpha}^{Gumbel}(u_1,u_2)+\alpha-1\right)}{u_1u_2\left[(-\log u_1)^{\alpha}+(-\log u_2)^{\alpha}\right]^{2\alpha-1/\alpha}}. \quad (16)$$

2.3.5 Frank Copula

The Frank copula is a symmetric copula in the Archimedean family. Its generator function is $\varphi(t)=-\log\left[(e^{-\alpha t}-1)/[e^{-\alpha}-1)\right]$ with $\alpha\in(-\infty,0)\cup(0,+\infty)$. The distribution function of the Frank copula is:

$$C_{\alpha}^{Frank}(u_1,u_2)=-\frac{1}{\alpha}\log\left(1+\frac{(e^{-\alpha u_1}-1)(e^{-\alpha u_2}-1)}{\exp(-\alpha)-1}\right),$$

and the density function is:

$$c_{\alpha}^{Frank}(u_1,u_2)=\frac{\alpha(1-e^{-\alpha})e^{-\alpha(u_1+u_2)}}{\left[(1-e^{-\alpha})-(1-e^{-\alpha u_1})(1-e^{-\alpha u_2})\right]^2}. \quad (17)$$

2.3.6 Mixture Copula

A mixture copula is still a copula, as shown by Li (2000), and is formed by the weighted average of two or more copulas. If C_1 and C_2 are copulas, and w_1 and w_2 are corresponding weights (with $w_1+w_2=1$), the mixture copula can be written as:

$$C(u_1,u_2;\alpha_1,\alpha_2)=w_1C_1(u_1,u_2;\alpha_1)+w_2C_2(u_1,u_2;\alpha_2),$$

where α_1 and α_{21} are parameters for C_1 and C_2. It is straightforward to obtain the density via (14):

$$c(u_1,u_2;\alpha_1,\alpha_2)=w_1c_1(u_1,u_2;\alpha_1)+w_2c_2(u_1,u_2;\alpha_2). \quad (18)$$

Mixture copulas can be used to combine different copulas. For instance, when C_1 has lower tail dependence but not upper tail dependence and the dependence structure for C_2 is reversed, the two-parameter mixture copula provides greater flexibility than either one-parameter copula. Furthermore, when C_2 is the survival copula of C_1, mixture copulas can be symmetric or asymmetric depending on the weights. A survival copula associated with a copula is defined as:

$$SC(u_1, u_2; \alpha) = \Pr(U_1 > u_1, U_2 > u_2) = u_1 + u_2 - 1 + C(1-u_1, 1-u_2; \alpha) ,$$

where α is the parameter. Survival copulas are the mirror images of corresponding copulas.

Mixture copulas have been widely used in financial research. Fortin and Kuzmics (2002) improved the fit achieved with common one-parameter copula models by means of mixtures for a set of European stocks. Patton (2006) proposed a symmetrized mixture between a Clayton copula and its survival copula with weights $w_1 = w_2 = 0.5$, which may provide a better fit. Chollete et al. (2005) also argued in favor of mixture copulas.

One should be cautious when interpreting the asymmetry studied in this paper. The asymmetry of the GJR-GARCH model for individual returns refers to its volatility, which reflects asymmetric response to past shocks. However, the distribution of the GJR-GARCH model is specified using a *t* distribution in this paper, which exhibits a symmetric pattern to help us focus on the asymmetry in the dependence structure. In contrast, the asymmetry of the bivariate distribution, sometimes called co-skewness, indicates that the first returns probability distribution may be skewed to the right (left) of the distribution of the second returns and represents a higher (lower) probability of extreme positive (negative) returns in both markets. This kind of asymmetry, which is the core of this paper, is dominated by copula functions and has been employed in literature on the higher-moment capital pricing model and the formation of portfolio strategies (Friend and Westerfield, 1980; Garcia and Tsafack, 2008).

2.4 Calculating Portfolio VaR

By definition, the VaR of two-asset portfolio returns with 5% level in the next period is the point in the one-day-ahead forecasted portfolio returns distribution at time t, D_t^f, that corresponds to the lower 5% tail. That is, $D_t^f(VaR) = 5\%$. CVaR is the expected value of the one-day-ahead forecasted portfolio returns $R_{p,t}^f$ at time t given that $R_{p,t}^f$ is worse than VaR. Mathematically, CVaR can be formulated as follows:

$$CVaR = E(R_{p,t}^f \mid R_{p,t}^f < VaR) .$$

Once the estimates of the bivariate copula-GJR-GARCH model described in the previous section are obtained at time t, we use a simulation procedure to calculate the 5% portfolio VaR and CVaR in the next period (i.e., $t+1$):

Step 1 Simulate j random samples of $\eta_{1,t}^{sim}$ and $\eta_{2,t}^{sim}$ from the specified copula distribution.

Step 2 Use the simulated random samples and the one-step-ahead forecasts of the GJR-GARCH volatility ($h_{1,t}^f$ and $h_{2,t}^f$) to generate the simulated residuals, defined as $\varepsilon_{1,t}^{sim} = \eta_{1,t}^{sim}\sqrt{h_{1,t}^f}$ and $\varepsilon_{2,t}^{sim} = \eta_{2,t}^{sim}\sqrt{h_{2,t}^f}$.

Step 3 Calculate the one-day-ahead forecast of the asset returns by $R_{1,t}^f = \hat{a}_0 + \hat{a}_1 R_{1,t-1} + \hat{a}_2 R_{2,t-1} + \varepsilon_{1,t}^{sim}$ and $R_{2,t}^f = \hat{b}_0 + \hat{b}_1 R_{1,t-1} + \hat{b}_2 R_{2,t-1} + \varepsilon_{2,t}^{sim}$ where $\hat{a}_i$ and $\hat{b}_i$ are estimates in the marginal GJR-GARCH models discussed in Section 2.2.

Step 4 Construct the distribution of the forecasts of the portfolio returns $R_{p,t}^f$ with specified portfolio weight w with respect to asset 1 by $R_{p,t}^f = wR_{1,t}^f + (1-w)R_{2,t}^f$.

Step 5 Measure VaR as the value of the 5% quantile in the distribution of the forecasts of portfolio returns ($R_{p,t}^f$). Measure CVaR as the mean value of the situations that $R_{p,t}^f$ exceeds the VaR.

Steps 1 to 5 are repeated until the end of the forecast period.

In contrast, the VaR of the DCC model is usually calculated by an analytical method that can be formulated as:

$$VaR = \mathbf{w}'\mathbf{R}_t^f + z_{5\%}\sqrt{\mathbf{w}'\mathbf{\Sigma}_{\mathbf{R},t}^f\mathbf{w}} ,$$

where $\mathbf{w}$ is the weight vector, $\mathbf{R}_t^f$ and $\mathbf{\Sigma}_{\mathbf{R},t}^f$ are the vector of the one-day-ahead return forecasts and the covariance matrix at time t , and $z_{5\%}$ is the left quantile of the normal distribution corresponding to 5% probability. However, keeping the same basis for comparison, we employ the simulation method. That is, the one-day-ahead forecasts of the portfolio returns are calculated by $R_{p,t}^f = \mathbf{w}'\mathbf{R}_t^f + \mathbf{z}\sqrt{\mathbf{w}'\mathbf{\Sigma}_{\mathbf{R},t}^f\mathbf{w}}$, where $\mathbf{z}$ is a j -dimensional random vector drawn from the normal distribution (the value of j equals the simulation times in the copula simulation method). Consequently, the VaR is measured as the value of the 5% quantile in the distribution of portfolio return forecasts ($R_{p,t}^f$) and the CVaR is calculated by the mean value of the situations that $R_{p,t}^f$ exceeds the VaR.

2.5 Model Validation

VaR models are only useful if they can be shown to be reasonably accurate. To do this, investors must systematically evaluate the validity of the VaR models through comparison between predicted and actual loss levels, a procedure known as backtesting. When a VaR model is built correctly, the probability of the number of observations falling outside the predicted VaR (known as exceptions) should be correspond to the nominal significance level.

When backtesting a VaR model, we could make two types of statistical errors: Type I errors occur when we reject the model which is correct, while type II errors occur when we fail to reject (i.e., incorrectly accept) the wrong model. Obviously, in risk management, it can be much more costly to incur type II errors, and therefore we should tolerate higher probability of Type I errors (α) in order to lower the probability of Type II errors (β). Moreover, a lower significance level α and small number of exceptions result in lower reliability. For example, choosing a significance level of 5% rather than 1%, we will observe more exception points and therefore have greater confidence in tests of model accuracy.

Kupiec (1995) developed a test statistic to determine whether the observed frequency of exceptions is consistent with the frequency of expected exceptions according to the VaR model and chosen confidence interval, defined by the tail points of the log-likelihood ratio:

$$LR_{UC} = -2\log\left[(1-p)^{T_0} p^{T_1}\right] + 2\log\left\{\left(1-\frac{T_1}{(T_0+T_1)}\right)^{T_0}\left(\frac{T_1}{(T_0+T_1)}\right)^{T_1}\right\} \sim \chi^2(1),$$

where T_0 is the number of losses not exceeding VaR and T_1 is the number of losses that exceed VaR. LR_{UC} is asymptotically distributed chi-square with 1 degree of freedom under the null hypothesis that p is the true probability of the occurrence of exceptions. The first term represents the maximal likelihood under the null hypothesis and the second term is the maximal likelihood for the observed data.

Kupiec's test only focuses on the frequency of exceptions and ignores the time dynamics of those exceptions. VaR models assume that exceptions should be independently distributed over time so that they should be unpredictable. If the exceptions exhibit some type of clustering, then the VaR model may fail under certain conditions, which could represent a potential problem. Following this logic, Christofferson (1998) developed a conditional coverage test (LR_{CC}) that incorporates the LR_{ind} statistic to test if today's occurrence of an exception is independent of what happened the previous day; it is defined by:

$$LR_{ind} = -2\log[(1-\pi)^{T_{00}+T_{10}}\pi^{T_{01}+T_{11}}] + 2\log\left(\prod_{i=0}^{1}(1-\pi_{i1})^{T_{i0}}\pi_{i1}^{T_{i1}}\right) \sim \chi^2(1),$$

where T_{ij} denotes the number of observations with state i followed by state j, $\pi_{i1} = T_{i1}/(T_{i0}+T_{i1})$ is the probability of observing an exception conditional in state i the previous day, and $i=1$ (or $j=1$) indicates that the VaR is exceeded. LR_{CC} is thus defined as $LR_{CC} = LR_{UC} + LR_{ind}$, which asymptotically distributed chi-square with 2 degrees of freedom. In this study, we employ the LR_{CC} test statistic since it is more robust, especially if the markets go through periods of calm and turbulence (Jorion, 2000).

3. Data and Results

Here we examine daily price indices of three major stock markets: S&P500, FTSE, and DAX obtained from Datastream. For the price series $p_{i,t}$ in market i, daily returns are calculated as $R_{i,t} = 100(\log p_{i,t} - \log p_{i,t-1})$. The sample period is from January 1991 to March 2006. The last 250 observations are used for a moving-window one-step-ahead out-of-sample accuracy test. That is, the length of the estimation period is fixed at 3725 observations, so that the start and end dates successively increase by one observation. For example, the first estimation period is 1991/01/01–2005/04/08, the second is 1991/01/02–2005/04/09, and so on.

Table 1 shows preliminary descriptive statistics. During the first estimation period (1991/01/01–2005/04/08), the average returns are positive for S&P500, FTSE, and DAX. The unconditional volatility, proxied by the sample standard deviation, indicates that the DAX was the most volatile and that S&P500 was the least volatile during this period. The skewness statistics suggest that returns are significantly negative-skewed, and the excess kurtosis statistics illustrate significant leptokurtosis for all markets (under normality). Consequently, the Jarque-Bera test statistics confirm that returns in all markets are significantly non-normal. In addition, test statistics for asymmetry in volatility proposed by Engle and Ng (1993) suggest a clear rejection of the null of no asymmetries in volatility of the return series.

Table 1. Descriptive Statistics for Daily Log Return of the Data Set

	S&P500	FTSE	DAX
N	3975	3975	3975
Mean	0.035	0.026	0.037
Standard deviation	0.992	1.011	1.403
Skewness	−0.098*	−0.118*	−0.288*
Excess Kurtosis	4.277*	3.466*	4.401*
JB test	3036.02*	1998.47*	3263.05*
EN test	66.05*	112.00*	87.03*

Notes: * denotes significance at the 5% level. JB test is the Jarque-Bera test statistic. EN test is the sign and size bias test statistic for asymmetry in volatility proposed by Engle and Ng (1993), which is asymptotically $\chi^2(3)$ -distributed under the null hypothesis of no asymmetric effects.

Table 2 shows the results from the copula-GJR-GARCH(1,1) models applied to the stock returns in each pair of markets. In the first column, "Marginal model 1" represents the market before the hyphen in the first row and "Marginal model 2" represents the market after the hyphen. Most of the estimates for the marginal models are significant, and all of them satisfy the stationarity condition of the GJR-GARCH model. The degrees of freedom for the t distributions for each index returns are all reasonably low, ranging from 7 to 13, and indicate that the use of t-distributed errors is more appropriate than normal distributed errors. They also imply the existence of the fourth moment in each market return series (since they exceed 4). Since the errors are assumed to be t distributed and the Ljung-Box Q statistics are based on a normality assumption, Smith (1985) suggests using the CDF of the t distribution with the estimated degrees of freedom to transform the original standardized residuals into cumulative probabilities and then map these probabilities into standard normal variables using the inverse Gaussian CDF function. The resulting series will follow a standard normal distribution. The Ljung-Box Q statistics on the transformed standardized residuals and squared transformed standardized residuals suggest the absence of serial correlation up to 10 lags at the

5% significance level. The non-significant Engle-Ng test statistics suggest the standardized residuals do not suffer from asymmetry in volatility, implying the GJR-GARCH *t*-distributed error models seem well specified. We do not eliminate the non-significant estimates here but do so in the forecasting procedure. The bottom half of Table 2 also reports the results for the seven copula dependence models and for the DCC model. All estimates are significant.

Table 3 reports the p-values of the LR_{CC} test statistics for each two-asset portfolio with different weights. At the 5% significance level, the DCC model and most copula models are accepted in all long positions across portfolios (the exceptions are *t* and Frank where 93% of long positions are accepted), while the DCC, Gumbel, and mixture Clayton models are accepted in all short positions across portfolios. Overall, the DCC, Gumbel, and mixture Clayton models are accepted for all portfolios and weights. We also perform the dynamic quantile test (Engle and Manganelli, 2004), which has higher statistical power, by investigating the regression of $Hits_t = I(R_t < VaR_t) - \alpha$ on lagged $Hits_t$ and lagged VaR_t (up to four time periods). The results (no reported but available upon request) show that almost all cases pass the backtesting procedure except (−0.5, −0.5) in the S&P500-DAX market pair at the 5% level. Since the results of the dynamic quantile test seems a looser criterion for selecting models than does LR_{CC}, we leave LR_{CC} as the backtesting method in this paper and evaluate "capital saving" via this stricter criterion.

Since the DCC, Gumbel, and mixture Clayton models are accepted in all cases, and the Gaussian, Clayton, and mixture Gumbel are accepted in all long position cases, what is the improvement of copula models compared with the DCC model? The Basel Committee (1996) mandates that banks can use a valid VaR model to set aside capital for market risk of the holding portfolios, implying that a valid model generating a lower value of VaR can reduce more capital charges for market risks. In addition, a lower CVaR represents lower expected loss if the actual portfolio returns are worse than the VaR. Based on these concepts, we apply the percentage difference of VaR and CVaR between each copula and the DCC models as measures of the relative performance. The percentage difference of VaR is defined as capital saving, and the percentage difference of CVaR is defined as potential loss reduction. Table 4 reports both measures for each copula models compared with the DCC model across different portfolios with different positions. Note that we only report the results for those 100% accepted models in Table 3.

Table 2. Parameter Estimates and Standard Errors for the Copula-GJR-GARCH Models

		S&P500-FTSE		S&P500-DAX		FTSE-DAX	
		Coeff.	Std. Err.	Coeff.	Std. Err.	Coeff.	Std. Err.
Marginal model 1	a_0	0.038*	(0.012)	0.039*	(0.012)	0.025**	(0.013)
	a_1	-0.007	(0.018)	0.001	(0.018)	0.025	(0.020)
	a_2	0.016	(0.016)	-0.002	(0.011)	-0.018	(0.014)
	α_0	0.006*	(0.001)	0.006*	(0.001)	0.009*	(0.002)
	α_1	0.002	(0.007)	0.002	(0.007)	0.008	(0.007)
	α_2	0.091*	(0.012)	0.090*	(0.012)	0.085*	(0.011)
	α_3	0.946*	(0.007)	0.947*	(0.007)	0.939*	(0.007)
	v_1	7.053*	(0.738)	7.051*	(0.738)	11.460*	(1.524)
	$Q(10)$	14.041 [0.171]		13.669 [0.189]		11.777 [0.300]	
	$Q^2(10)$	12.339 [0.263]		12.725 [0.239]		13.989 [0.174]	
	EN test	4.864		3.637		4.128	
Marginal model 2	b_0	0.013	(0.012)	0.043*	(0.015)	0.052*	(0.016)
	b_1	-0.104*	(0.017)	-0.124*	(0.017)	-0.088*	(0.020)
	b_2	0.301*	(0.016)	0.444*	(0.019)	0.183*	(0.023)
	β_0	0.008*	(0.002)	0.010*	(0.003)	0.016*	(0.004)
	β_1	0.007	(0.006)	0.052*	(0.011)	0.043*	(0.011)
	β_2	0.087*	(0.012)	0.047*	(0.014)	0.068*	(0.014)
	β_3	0.940*	(0.007)	0.920*	(0.008)	0.913*	(0.009)
	v_2	12.322*	(1.763)	8.330*	(0.679)	8.235*	(0.692)
	$Q(10)$	12.362 [0.262]		8.464 [0.584]		10.972 [0.360]	
	$Q^2(10)$	11.679 [0.307]		5.321 [0.869]		15.007 [0.132]	
	EN test	0.613		4.646		6.655	
DCC	θ_1	0.006*	(0.001)	0.011*	(0.001)	0.034*	(0.001)
	θ_2	0.988*	(0.003)	0.988*	(0.001)	0.960*	(0.001)
Gaussian	ρ_n	0.390*	(0.012)	0.363*	(0.012)	0.588*	(0.008)
t	ρ_t	0.388*	(0.014)	0.364*	(0.014)	0.596*	(0.010)
	v	13.154*	(3.243)	12.142*	(2.819)	7.774*	(1.105)
Clayton	θ_c	0.458*	(0.023)	0.391*	(0.021)	0.950*	(0.027)
Gumbel	θ_g	1.313*	(0.016)	1.298*	(0.016)	1.619*	(0.019)
Frank	θ_f	2.399*	(0.101)	2.267*	(0.099)	4.352*	(0.109)
Mixture Clayton	θ_{mc}	0.582*	(0.074)	0.421*	(0.075)	1.070*	(0.072)
	θ_{mic}	0.738*	(0.090)	0.800*	(0.132)	1.705*	(0.195)
	w_{mc}	0.503*	(0.053)	0.510*	(0.066)	0.637*	(0.032)
Mixture Gumbel	θ_{mg}	1.366*	(0.030)	1.319*	(0.122)	2.070*	(0.083)
	θ_{mig}	1.296*	(0.034)	1.295*	(0.239)	1.537*	(0.034)
	w_{mg}	0.537*	(0.055)	0.669*	(0.159)	0.335*	(0.037)

Notes: * and ** denote the 5% and 10% levels. $Q(10)$ and $Q^2(10)$ are Ljung-Box autocorrelation test statistic for transformed standardized residuals and squared standardized residuals up to 10 lags. Ljung-Box Q statistic p-values are in brackets. EN test is the sign and size bias test statistics for asymmetry in volatility proposed by Engle and Ng (1993), which is asymptotically $\chi^2(3)$ -distributed under the null hypothesis of no asymmetric effects.

Table 3. P-Values for LR_{cc} Accuracy Tests

	Symmetric dependence structure				Asymmetric dependence structure			
	DCC	**Gaussian**	***t***	**Frank**	**Clayton**	**Gumbel**	**Mixture Clayton**	**Mixture Gumbel**
S&P500-FTSE								
Long position								
(0.5, 0.5)	0.545	0.097	0.181	0.290	0.539	0.539	0.539	0.539
(0.25, 0.75)	0.404	0.097	0.097	0.181	0.396	0.396	0.396	0.396
(0.75, 0.25)	0.290	0.097	0.097	0.097	0.483	0.886	0.886	0.483
(1.25, −0.25)	0.342	0.290	0.097	0.043*	0.342	0.342	0.342	0.342
(−0.25, 1.25)	0.290	0.097	0.097	0.181	0.290	0.290	0.290	0.290
Short position								
(−0.5, −0.5)	0.496	0.097	0.043*	0.097	0.457	0.918	0.918	0.918
(−0.25, −0.75)	0.539	0.290	0.290	0.290	0.133	0.207	0.298	0.207
(−0.75, −0.25)	0.404	0.043*	0.097	0.097	0.496	0.404	0.496	0.496
(−1.25, 0.25)	0.181	0.016*	0.004*	0.004*	0.097	0.181	0.181	0.181
(0.25, −1.25)	0.483	0.404	0.404	0.290	0.539	0.298	0.396	0.396
S&P500-−DAX								
Long position								
(0.5, 0.5)	0.545	0.181	0.097	0.404	0.267	0.126	0.298	0.396
(0.25, 0.75)	0.858	0.530	0.530	0.530	0.191	0.191	0.191	0.191
(0.75, 0.25)	0.404	0.097	0.097	0.097	0.396	0.396	0.396	0.396
(1.25, −0.25)	0.530	0.088	0.034*	0.088	0.530	0.530	0.530	0.530
(−0.25, 1.25)	0.918	0.342	0.530	0.342	0.776	0.886	0.776	0.776
Short position								
(−0.5, −0.5)	0.886	0.097	0.097	0.181	0.457	0.621	0.621	0.621
(−0.25, −0.75)	0.918	0.097	0.097	0.097	0.021*	0.120	0.120	0.031*
(−0.75, −0.25)	0.404	0.097	0.043*	0.097	0.545	0.496	0.496	0.496
(−1.25, 0.25)	0.290	0.001*	0.004*	0.004*	0.016*	0.290	0.181	0.181
(0.25, −1.25)	0.284	0.192	0.192	0.192	0.284	0.124	0.124	0.124
FTSE-−DAX								
Long position								
(0.5, 0.5)	0.717	0.530	0.190	0.530	0.190	0.717	0.344	0.344
(0.25, 0.75)	0.717	0.342	0.342	0.342	0.191	0.191	0.191	0.126
(0.75, 0.25)	0.717	0.530	0.530	0.530	0.406	0.134	0.406	0.406
(1.25, −0.25)	0.190	0.097	0.097	0.097	0.190	0.181	0.190	0.190
(−0.25, 1.25)	0.776	0.342	0.342	0.342	0.776	0.776	0.776	0.776
Short position								
(−0.5, −0.5)	0.284	0.545	0.545	0.545	0.191	0.436	0.436	0.436
(−0.25, −0.75)	0.367	0.192	0.113	0.192	0.120	0.120	0.120	0.120
(−0.75, −0.25)	0.483	0.404	0.404	0.404	0.133	0.298	0.298	0.396
(−1.25, 0.25)	0.496	0.016*	0.016*	0.016*	0.097	0.496	0.181	0.181
(0.25, −1.25)	0.072	0.290	0.290	0.290	0.072	0.101	0.101	0.101
Long position	100%	100%	93%	93%	100%	100%	100%	100%
Short position	100%	73%	67%	80%	87%	100%	100%	93%
Overall	100%	87%	80%	87%	93%	100%	100%	97%

Notes: * denotes significance at the 5% level.

Table 4. Performance Comparison

	Symmetric dependence structure				Asymmetric dependence structure							
	DCC		Gaussian		Clayton		Gumbel		Mixture Clayton		Mixture Gumbel	
	Average VaR	Average CVaR	Average VaR	Average CVaR	Average VaR	Average CVaR	Average VaR	Average CVaR	Average VaR	Average CVaR	Average VaR	Average CVaR
S&P500-FTSE												
Long position												
(0.5, 0.5)	-0.851	-1.072	-0.958	-1.272	-0.777	-0.981	-0.764	-0.948	-0.776	-0.968	-0.777	-0.970
	—	—	(12.57)	(18.66)	(-8.70)	(-8.49)	(-10.22)	(-11.57)	(-8.81)	(-9.70)	(-8.70)	(-9.51)
(0.25, 0.75)	-0.855	-1.075	-0.941	-1.234	-0.771	-0.963	-0.765	-0.943	-0.773	-0.956	-0.773	-0.957
	—	—	(10.06)	(14.79)	(-9.82)	(-10.42)	(-10.53)	(-12.28)	(-9.59)	(-11.07)	(-9.59)	(-10.98)
(0.75, 0.25)	-0.936	-1.181	-1.070	-1.453	-0.873	-1.102	-0.873	-1.094	-0.879	-1.101	-0.878	-1.101
	—	—	(14.32)	(23.03)	(-6.73)	(-6.69)	(-6.73)	(-7.37)	(-6.09)	(-6.77)	(-6.20)	(-6.77)
(1.25, −0.25)	-1.285	-1.623	-1.485	-2.049	-1.256	-1.579	-1.284	-1.631	-1.261	-1.596	-1.260	-1.595
	—	—	(15.56)	(26.25)	(-2.26)	(-2.71)	(-0.08)	(0.49)	(-1.87)	(-1.66)	(-1.95)	(-1.73)
(−0.25, 1.25)	-1.117	-1.400	-1.207	-1.581	-1.054	-1.293	-1.074	-1.333	-1.053	-1.302	-1.052	-1.302
	—	—	(8.06)	(12.93)	(-5.64)	(-7.64)	(-3.85)	(-4.79)	(-5.73)	(-7.00)	(-5.82)	(-7.00)
Sub-average	—	—	12.1%	19.1%	-6.6%	-7.2%	-6.3%	-7.1%	-6.4%	-7.2%	-6.5%	-7.2%
Short position												
(−0.5, −0.5)	-0.890	-1.112	—	—	—	—	-0.829	-1.037	-0.822	-1.020	—	—
	—	—	—	—	—	—	(-6.85)	(-6.74)	(-7.64)	(-8.27)	—	—
(−0.25, −0.75)	-0.878	-1.098	—	—	—	—	-0.804	-0.999	-0.801	-0.989	—	—
	—	—	—	—	—	—	(-8.43)	(-9.02)	(-8.77)	(-9.93)	—	—
(−0.75, −0.25)	-0.992	-1.237	—	—	—	—	-0.937	-1.169	-0.938	-1.165	—	—
	—	—	—	—	—	—	(-5.54)	(-5.50)	(-5.44)	(-5.82)	—	—
(−1.25, 0.25)	-1.374	-1.712	—	—	—	—	-1.330	-1.651	-1.345	-1.676	—	—
	—	—	—	—	—	—	(-3.20)	(-3.56)	(-2.11)	(-2.10)	—	—
(0.25, −1.25)	-1.107	-1.390	—	—	—	—	-1.025	-1.264	-1.038	-1.283	—	—
	—	—	—	—	—	—	(-7.41)	(-9.06)	(-6.23)	(-7.70)	—	—
Sub-average	—	—	—	—	—	—	-6.3%	-6.8%	-6.0%	-6.8%	—	—
S&P500-DAX												
Long position												
(0.5, 0.5)	-0.990	-1.252	-1.116	-1.490	-0.895	-1.138	-0.888	-1.109	-0.896	-1.123	-0.899	-1.127
	—	—	(12.73)	(19.01)	(-9.60)	(-9.11)	(-10.30)	(-11.42)	(-9.49)	(-10.30)	(-9.19)	(-9.98)
(0.25, 0.75)	-1.097	-1.387	-1.228	-1.646	-1.000	-1.262	-1.004	-1.254	-1.007	-1.259	-1.008	-1.261
	—	—	(11.94)	(18.67)	(-8.84)	(-9.01)	(-8.48)	(-9.59)	(-8.20)	(-9.23)	(-8.11)	(-9.08)
(0.75, 0.25)	-0.986	-1.246	-1.123	-1.521	-0.907	-1.152	-0.908	-1.139	-0.913	-1.147	-0.914	-1.149
	—	—	(13.89)	(22.07)	(-8.01)	(-7.54)	(-7.91)	(-8.59)	(-7.40)	(-7.95)	(-7.30)	(-7.78)
(1.25, −0.25)	-1.272	-1.604	-1.481	-2.037	-1.274	-1.600	-1.291	-1.641	-1.275	-1.614	-1.270	-1.612
	—	—	(16.43)	(27.00)	(0.16)	(-0.25)	(1.49)	(2.31)	(0.24)	(0.62)	(-0.16)	(0.50)
(−0.25, 1.25)	-1.523	-1.922	-1.713	-2.312	-1.468	-1.832	-1.488	-1.873	-1.473	-1.849	-1.469	-1.845
	—	—	(12.48)	(20.29)	(-3.61)	(-4.68)	(-2.30)	(-2.55)	(-3.28)	(-3.80)	(-3.55)	(-4.01)
Sub-average	—	—	13.5%	21.4%	-6.0%	-6.1%	-5.5%	-6.0%	-5.6%	-6.1%	-5.7%	-6.1%

	Symmetric dependence structure				Asymmetric dependence structure							
	DCC		Gaussian		Clayton		Gumbel		Mixture Clayton		Mixture Gumbel	
	Average VaR	Average CVaR	Average VaR	Average CVaR	Average VaR	Average CVaR	Average VaR	Average CVaR	Average VaR	Average CVaR	Average VaR	Average CVaR
Short position												
(−0.5, −0.5)	-1.072	-1.334	—	—	—	—	-1.000	-1.250	-0.992	-1.232	—	—
	—	—	—	—	—	—	(-6.72)	(-6.30)	(-7.46)	(-7.65)	—	—
(−0.25, −0.75)	-1.184	-1.474	—	—	—	—	-1.102	-1.367	-1.101	-1.360	—	—
	—	—	—	—	—	—	(-6.93)	(-7.26)	(-7.01)	(-7.73)	—	—
(−0.75, −0.25)	-1.064	-1.324	—	—	—	—	-1.001	-1.250	-0.999	-1.241	—	—
	—	—	—	—	—	—	(-5.92)	(-5.59)	(-6.11)	(-6.27)	—	—
(−1.25.0.25)	-1.342	-1.674	—	—	—	—	-1.313	-1.633	-1.328	-1.657	—	—
	—	—	—	—	—	—	(-2.16)	(-2.45)	(-1.04)	(-1.02)	—	—
(0.25, −1.25)	-1.618	-2.017	—	—	—	—	-1.538	-1.895	-1.553	-1.919	—	—
	—	—	—	—	—	—	(-4.94)	(-6.05)	(-4.02)	(-4.86)	—	—
Sub-average	—	—	—	—	—	—	-5.3%	-5.5%	-5.1%	-5.5%	—	—
FTSE-DAX												
Long position												
(0.5, 0.5)	-1.097	-1.386	-1.172	-1.559	-0.954	-1.194	-0.920	-1.138	-0.950	-1.178	-0.950	-1.182
	—	—	(6.84)	(12.48)	(-13.04)	(-13.85)	(-16.13)	(-17.89)	(-13.40)	(-15.01)	(-13.40)	(-14.72)
(0.25, 0.75)	-1.215	-1.536	-1.329	-1.785	-1.081	-1.351	-1.071	-1.337	-1.084	-1.352	-1.081	-1.351
	—	—	(9.38)	(16.21)	(-11.03)	(-12.04)	(-11.85)	(-12.96)	(-10.78)	(-11.98)	(-11.03)	(-12.04)
(0.75, 0.25)	-1.018	-1.285	-1.084	-1.432	-0.885	-1.101	-0.860	-1.059	-0.882	-1.090	-0.881	-1.092
	—	—	(6.48)	(11.44)	(-13.06)	(-14.32)	(-15.52)	(-17.59)	(-13.36)	(-15.18)	(-13.46)	(-15.02)
(1.25, −0.25)	-1.018	-1.281	-1.171	-1.552	-1.005	-1.228	-1.055	-1.325	-1.007	-1.251	-1.003	-1.248
	—	—	(15.03)	(21.16)	(-1.28)	(-4.14)	(3.63)	(3.43)	(-1.08)	(-2.34)	(-1.47)	(-2.58)
(−0.25.1.25)	-1.532	-1.938	-1.770	-2.405	-1.468	-1.819	-1.524	-1.931	-1.480	-1.856	-1.474	-1.850
	—	—	(15.54)	(24.10)	(-4.18)	(-6.14)	(-0.52)	(-0.36)	(-3.39)	(-4.23)	(-3.79)	(-4.54)
Sub-average	—	—	10.7%	17.1%	-8.5%	-10.1%	-8.1%	-9.1%	-8.4%	-9.8%	-8.6%	-9.8%
Short position												
(−0.5, −0.5)	-1.177	-1.466	—	—	—	—	-1.040	-1.284	-1.014	-1.238	—	—
	—	—	—	—	—	—	(-11.64)	(-12.41)	(-13.85)	(-15.55)	—	—
(−0.25, −0.75)	-1.312	-1.633	—	—	—	—	-1.185	-1.459	-1.178	-1.441	—	—
	—	—	—	—	—	—	(-9.68)	(-10.66)	(-10.21)	(-11.76)	—	—
(−0.75, −0.25)	-1.082	-1.349	—	—	—	—	-0.954	-1.173	-0.935	-1.139	—	—
	—	—	—	—	—	—	(-11.83)	(-13.05)	(-13.59)	(-15.57)	—	—
(−1.25, 0.25)	-1.049	-1.312	—	—	—	—	-1.024	-1.257	-1.056	-1.325	—	—
	—	—	—	—	—	—	(-2.38)	(-4.19)	(0.67)	(0.99)	—	—
(0.25, −1.25)	-1.661	-2.067	—	—	—	—	-1.596	-1.956	-1.633	-2.025	—	—
	—	—	—	—	—	—	(-3.91)	(-5.37)	(-1.69)	(-2.03)	—	—
Sub-average	—	—	—	—	—	—	-7.9%	-9.1%	-7.7%	-8.8	—	—
Long position	—	—	12.1 %	19.2 %	-7.0 %	-7.8 %	-6.6 %	-7.4 %	-6.8 %	-7.7 %	-6.9 %	-7.7 %
Short Position	—	—	—	—	—	—	-6.5 %	-7.1 %	-6.3 %	-7.0 %	—	—
Overall	—	—	—	—	—	—	-6.6 %	-7.3 %	-6.6 %	-7.4 %	—	—

Notes: Percentage differences of VaR and CVaR of each copula model and the DCC model are in parentheses.

Based on the criterion of capital saving (i.e., percentage difference of VaR), Clayton is performs best in the S&P500-FTSE and S&P500-DAX portfolios for the long position, saving on average 6.6% and 6.0% capital charges of market risk compared with the DCC model, while the mixture Gumbel is performs best in the FTSE-DAX portfolio, saving on average 8.6% capital charges. For short positions, the Gumbel model saves on average 6.3% capital charges in the S&P500-FTSE, 5.3% in the S&P500-DAX, and 7.9% in the FTSE-DAX portfolio. Considering both long and short positions, the Gumbel and mixture Clayton models save on average 6.6% capital charges. Averaging across portfolios with different weights, the Clayton and Gumbel perform best for long and short positions, saving 7.0% and 6.5% capital charges. Overall, the performances of the Gumbel and mixture Clayton both outperform the DCC, saving 6.6% capital charges. Taking the numerical example mentioned in Section 1 as benchmark, the models with asymmetric dependence structure could deliver profits of at least three million dollars per year.

According to the criterion of potential loss reduction (i.e., percentage difference of CVaR), the Clayton, Gumbel, mixture Clayton, and mixture Gumbel modes all have about 7% lower potential loss in the S&P500-FTSE compared with the DCC for long positions, 6% in the S&P500-DAX, and 10% in the FTSE-DAX portfolios. For short positions, the Gumbel and mixture Clayton models have 6.8% lower potential loss compared with the DCC in the S&P500-FTSE, 5.5% in the S&P500-DAX, and 9% in the FTSE-DAX portfolios. Considering both long and short positions, the Gumbel and mixture Clayton models have about 7% lower potential loss. Averaging across portfolios with different weights, the Clayton and Gumbel still perform best for long and short positions (7.8% and 7.1% lower potential loss). Overall, the Gumbel and mixture Clayton models have similar potential loss reduction. According to these results, we find that the asymmetric copula models indeed outperform the symmetric DCC and symmetric copula models, implying the importance of asymmetric dependence structure.

4. Conclusions

It has been widely accepted that stock returns are asymmetrically dependent across markets. To investigate the importance of asymmetric dependence structures in VaR calculations, we introduce copulas. This work shows that many copula models are acceptable and satisfy backtesting validation, as does the competing DCC model. However, under similar levels of acceptability, the copula models with asymmetric dependence structures can save capital charges for market risks and reduce potential loss compared with the DCC model. Since copula models with asymmetric dependence structures also work better than symmetric models based on the two criteria we propose, the results also imply that asymmetric dependence structures are of great importance in improving VaR calculations and deliver potential economic values. Since banks or firms always have an incentive to take higher risks and thus to choose a model which generates lower economic capital charges for capital conservation, the capital saving resulting from one model over the other can be a measure of that incentive and could be applied to select models from an economic viewpoint, which is an important contribution of this paper.

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